Chapter 8

Zweier I-Convergent Double Sequence Spaces

8.1 Introduction

At the initial stage the notion of I-convergence was introduced by Kostyrko, Šalát and Wilczyński[48]. Later on it was studied by Šalát, Tripathy and Ziman[65], Demirci [10] and many others. I-convergence is a generalization of Statistical Convergence.

Now we have a list of some basic definitions used in the chapter:

Definition 8.1. A double sequence of complex numbers is defined as a function $x : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$. We denote a double sequence as (x_{ij}), where the two subscripts run through the sequence of natural numbers independent of each other. A number $a \in \mathbb{C}$ is called a double limit of a double sequence (x_{ij}) if for every $\epsilon > 0$ there exists some $N = N(\epsilon) \in \mathbb{N}$ such that

$$|(x_{ij}) - a| < \epsilon, \qquad for\ all\ i, j \geq N \ \ (see\ [6, 7, 8])$$

Definition 8.2. A double sequence $(x_{ij}) \in \omega$ is said to be I-convergent to a number L if for every $\epsilon > 0$,

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I.$$

In this case we write $I - \lim x_{ij} = L$.

Definition 8.3. A double sequence $(x_{ij}) \in \omega$ is said to be I-null if L = 0. In this case we write

$$I - \lim x_{ij} = 0.$$

Definition 8.4. A double sequence $(x_{ij}) \in \omega$ is said to be I-cauchy if for

"Example is the school of mankind, and they will learn at no other."-Edmund Burke every $\epsilon > 0$ there exist numbers m = m(ϵ), n= n(ϵ) such that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{mn}| \geq \epsilon\} \in I.$$

Definition 8.5. A double sequence $(x_{ij}) \in \omega$ is said to be I-bounded if there exists $M > 0$ such that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}| > M\}.$$

Definition 8.6. A double sequence space E is said to be solid or normal if $(x_{ij}) \in$ E implies $(\alpha_{ij} x_{ij}) \in$ E for all sequence of scalars (α_{ij}) with $|\alpha_{ij}| < 1$ for all (i, j) $\in \mathbb{N} \times \mathbb{N}$.

Definition 8.7. A double sequence space E is said to be monotone if it contains the canonical preimages of its stepspaces.

Definition 8.8. A double sequence space E is said to be convergence free if $(y_{ij}) \in E$ whenever $(x_{ij}) \in E$ and $x_{ij} = 0$ implies $y_{ij} = 0$.

Definition 8.9. A double sequence space E is said to be a sequence algebra if $(x_{ij}.y_{ij}) \in E$ whenever $(x_{ij}), (y_{ij}) \in E$.

Definition 8.10. A double sequence space E is said to be symmetric if $(x_{ij}) \in E$ implies $(x_{\pi(ij)}) \in E$, where π is a permutation on $\mathbb{N} \times \mathbb{N}$.

In this Chapter we introduce the following classes of sequence space:

$${}_2\mathcal{Z}^I = \{x = (x_{ij}) \in {}_2\omega : I - \lim Z^p x = L \text{ for some L} \in \mathbb{C}\ \}$$

$${}_2\mathcal{Z}_0^I = \{x = (x_{ij}) \in {}_2\omega : I - \lim Z^p x = 0\}$$

$${}_2\mathcal{Z}_\infty^I = \{x = (x_{ij}) \in {}_2\omega : \{(i,j) \in \mathbb{N} \times \mathbb{N} : \text{there exist } M > 0, |Z^p x| \geq M\} \in I\}$$

$${}_2\mathcal{Z}_\infty = \{x = (x_{ij}) \in {}_2\omega : \sup_{i,j} |Z^p x| < \infty\}$$

We also denote the multiplier double sequence spaces as

$$ {}_2m^I_{\mathcal{Z}} = {}_2\mathcal{Z}_\infty \cap {}_2\mathcal{Z}^I \quad and \quad {}_2m^I_{\mathcal{Z}_0} = {}_2\mathcal{Z}_\infty \cap {}_2\mathcal{Z}^I_0. $$

8.2 Main Results

Theorem 8.2.1. The classes of sequences ${}_2\mathcal{Z}^I$, ${}_2\mathcal{Z}^I_0$, ${}_2m^I_{\mathcal{Z}}$ and ${}_2m^I_{\mathcal{Z}_0}$ are linear spaces.

Proof. We shall prove the result for the space ${}_2\mathcal{Z}^I$. The proof for the other spaces will follow similarly. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}^I$ and let α, β be scalars. Then

$$ I - \lim |x_{ij} - L_1| = 0, \text{ for some } L_1 \in \mathbb{C}; $$

$$ I - \lim |y_{ij} - L_2| = 0, \text{ for some } L_2 \in \mathbb{C}. $$

That is for a given $\epsilon > 0$, we have

$$ A_1 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L_1| > \frac{\epsilon}{2}\} \in I, \quad [8.1] $$

$$ A_2 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - L_2| > \frac{\epsilon}{2}\} \in I. \quad [8.2] $$

We have

$$ |(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2)| \leq |\alpha|(|x_{ij} - L_1|) + |\beta|(|y_{ij} - L_2|) $$
$$ \leq |x_{ij} - L_1| + |y_{ij} - L_2|. $$

Now, by [8.1] and [8.2],

$$ \{(i,j) \in \mathbb{N} \times \mathbb{N} : |(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2)| > \epsilon\} \subset A_1 \cup A_2. $$

Therefore $(\alpha x_{ij} + \beta y_{ij}) \in {}_2\mathcal{Z}^I$. Hence ${}_2\mathcal{Z}^I$ is a linear space.

We state the following result without proof in view of Theorem 2.1.

Theorem 8.2.2. The spaces ${}_2m^I_{\mathcal{Z}}$ and ${}_2m^I_{\mathcal{Z}_0}$ are normed linear spaces, normed by

$$||x_{ij}||_* = \sup_{i,j} |x_{ij}|. \qquad [8.3]$$

Theorem 8.2.3. A sequence $x = (x_{ij}) \in {}_2m^I_{\mathcal{Z}}$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon = (m, n) \in \mathbb{N} \times \mathbb{N}$ such that

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{N_\epsilon}| < \epsilon\} \in {}_2m^I_{\mathcal{Z}} \qquad [8.4]$$

Proof. Suppose that $L = I - \lim x$. Then

$$B_\epsilon = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| < \frac{\epsilon}{2}\} \in {}_2m^I_{\mathcal{Z}} \text{ for all } \epsilon > 0.$$

Fix an $N_\epsilon = (m, n) \in B_\epsilon$. Then we have

$$|x_{N_\epsilon} - x_{ij}| \leq |x_{N_\epsilon} - L| + |L - x_{ij}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $(i, j) \in B_\epsilon$. Hence
$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{N_\epsilon}| < \epsilon\} \in {}_2m^I_{\mathcal{Z}}$.

Conversely, suppose that $\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{N_\epsilon}| < \epsilon\} \in {}_2m^I_{\mathcal{Z}}$. That is

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_k - x_{N_\epsilon}| < \epsilon\} \in {}_2m^I_{\mathcal{Z}}$$

for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{(i, j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in {}_2m^I_{\mathcal{Z}} \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in {}_2m^I_{\mathcal{Z}}$ as well as $C_{\frac{\epsilon}{2}} \in {}_2m^I_{\mathcal{Z}}$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in {}_2m^I_{\mathcal{Z}}$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \in J\} \in {}_2m^I_{\mathcal{Z}}$$

that is

$$diamJ \leq diamJ_\epsilon$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diamI_k \leq \frac{1}{2}diamI_{k-1}$ for (k=2,3,4,.....) and $\{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \in I_k\} \in {}_2m^I_{\mathcal{Z}}$ for (k=1,2,3,4,......). Then there exists a $\xi \in \cap I_k$ where $(i,j) \in \mathbb{N} \times \mathbb{N}$ such that $\xi = I - \lim x$, that is $L = I - \lim x$.

Theorem 8.2.4. Let I be an admissible ideal. Then the following are equivalent.

(a) $(x_{ij}) \in {}_2\mathcal{Z}^I$;

(b) there exists $(y_{ij}) \in {}_2\mathcal{Z}$ such that $x_{ij} = y_{ij}$, for a.a.k.r.I;

(c) there exists$(y_{ij}) \in {}_2\mathcal{Z}$ and $(z_{ij}) \in {}_2\mathcal{Z}^I_0$ such that $x_{ij} = y_{ij} + z_{ij}$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$ and

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - L| \geq \epsilon\} \in I;$$

(d) there exists a subset $K = \{k_1 < k_2....\}$ of $\mathbb{N}$ such that $K \in \pounds(I)$ and $\lim_{n\to\infty} |x_{k_n} - L| = 0$.

Proof. (a) implies (b). Let $(x_{ij}) \in {}_2\mathcal{Z}^I$. Then there exists $L \in \mathbb{C}$ such that

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \in I.$$

Let (m_t, n_t) be an increasing sequence with $(m_t, n_t) \in \mathbb{N} \times \mathbb{N}$ such that

$$\{(i,j) \leq (m_t, n_t) : |x_{ij} - L| \geq \frac{1}{t}\} \in I.$$

Define a sequence (y_{ij}) as

$$y_{ij} = x_{ij}, \text{ for all } (i,j) \leq (m_1, n_1).$$

For $(m_t, n_t) < (i,j) \leq (m_{t+1}, n_{t+1}) \quad for \quad t \in \mathbb{N}$.

$$y_{ij} = \begin{cases} x_{ij}, & \text{if } |x_{ij} - L| < t^{-1}, \\ L, & \text{otherwise.} \end{cases}$$

Then $(y_{ij}) \in {}_2\mathcal{Z}$ and form the following inclusion

$$\{(i,j) \leq (m_t, n_t) : x_{ij} \neq y_{ij}\} \subseteq \{(i,j) \leq (m_t, n_t) : |x_{ij} - L| \geq \epsilon\} \in I.$$

We get $x_{ij} = y_{ij}$, for a.a.k.r.I.

(b) implies (c). For $(x_{ij}) \in {}_2\mathcal{Z}^I$, there exists $(y_{ij}) \in {}_2\mathcal{Z}$ such that $x_{ij} = y_{ij}$, for a.a.k.r.I. Let $K = \{(i,j) \in \mathbb{N} \times \mathbb{N} : x_{ij} \neq y_{ij}\}$, then $K \in I$. Define a sequence (z_{ij}) as

$$z_{ij} = \begin{cases} x_{ij} - y_{ij}, & \text{if } (i,j) \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Then $z_{ij} \in {}_2\mathcal{Z}_0^I$ and $y_{ij} \in {}_2\mathcal{Z}$.

(c) implies (d). Let $P_1 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |z_{ij}| \geq \epsilon\} \in I$ and

$$K = P_1^c = \{(i_1, j_1) < (i_2, j_2) < ...\} \in \pounds(I).$$

Then we have $\lim_{n \to \infty} |x_{(i_n, j_n)} - L| = 0$.

(d) implies (a). Let $K = \{(i_1, j_1) < (i_2, j_2) < ...\} \in \pounds(I)$ and $\lim_{n \to \infty} |x_{(i_n, j_n)} - L| = 0$. Then for any $\epsilon > 0$, and Lemma 1.17, we have

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \epsilon\} \subseteq K^c \cup \{(i,j) \in K : |x_{ij} - L| \geq \epsilon\}.$$

Thus $(x_{ij}) \in {}_2\mathcal{Z}^I$.

Theorem 8.2.5. The inclusions ${}_2\mathcal{Z}_0^I \subset {}_2\mathcal{Z}^I \subset {}_2\mathcal{Z}_\infty^I$ hold and are proper.

Proof. Let $(x_{ij}) \in {}_2\mathcal{Z}^I$. Then there exists L $\in \mathbb{C}$ such that

$$I - \lim |x_{ij} - L| = 0$$

We have $|x_{ij}| \leq \frac{1}{2}|x_{ij} - L| + \frac{1}{2}|L|$. Taking the supremum over (i, j) on both sides we get $(x_{ij}) \in {}_2\mathcal{Z}_\infty^I$. The inclusion ${}_2\mathcal{Z}_0^I \subset {}_2\mathcal{Z}^I$ is obvious. The strict inclusion is also trivial.

Theorem 8.2.6. The function $\hbar : {}_2m_\mathcal{Z}^I \to \mathbb{R}$ is the Lipschitz function, where ${}_2m_\mathcal{Z}^I = {}_2\mathcal{Z}^I \cap {}_2\mathcal{Z}_\infty$, and hence uniformly continuous.

Proof. Let $x, y \in {}_2m_\mathcal{Z}^I$, $x \neq y$. Then the sets

$$A_x = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| \geq ||x - y||_*\} \in I,$$

$$A_y = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)| \geq ||x - y||_*\} \in I.$$

Thus the sets,

$$B_x = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - \hbar(x)| < ||x - y||_*\} \in {}_2m_\mathcal{Z}^I,$$

$$B_y = \{(i, j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - \hbar(y)| < ||x - y||_*\} \in {}_2m_\mathcal{Z}^I.$$

Hence also $B = B_x \cap B_y \in {}_2m_\mathcal{Z}^I$, so that $B \neq \phi$. Now taking (i,j) in B,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_{ij}| + |x_{ij} - y_{ij}| + |y - \hbar(y)| \leq 3||x - y||_*.$$

Thus $\hbar$ is a Lipschitz function. For ${}_2m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 8.2.7. If $x, y \in {}_2m_\mathcal{Z}^I$, then $(x.y) \in {}_2m_\mathcal{Z}^I$ and
$\hbar(xy) = \hbar(x)\hbar(y)$.

Proof. For $\epsilon > 0$

$$B_x = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x - \hbar(x)| < \epsilon\} \in {}_2m_{\mathcal{Z}}^I,$$

$$B_y = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y - \hbar(y)| < \epsilon\} \in {}_2m_{\mathcal{Z}}^I.$$

Now,

$$|x.y - \hbar(x)\hbar(y)| = |x.y - x\hbar(y) + x\hbar(y) - \hbar(x)\hbar(y)|$$

$$\leq |x||y - \hbar(y)| + |\hbar(y)||x - \hbar(x)| \qquad [8.5]$$

As ${}_2m_{\mathcal{Z}}^I \subseteq {}_2\mathcal{Z}_\infty$, there exists an $M \in \mathbb{R}$ such that $\hbar|x| < M$ and $|\hbar(y)| < M$. Using eqn[8.5] we get

$$|x.y - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all $(i,j) \in B_x \cap B_y \in {}_2m_{\mathcal{Z}}^I$. Hence $(x.y) \in {}_2m_{\mathcal{Z}}^I$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For ${}_2m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 8.2.8. The spaces ${}_2\mathcal{Z}_0^I$ and ${}_2m_{\mathcal{Z}_0}^I$ are solid and monotone .

Proof. We shall prove the result for ${}_2\mathcal{Z}_0^I$. Let $(x_{ij}) \in \mathcal{Z}_0^I$. Then

$$I - \lim_k |x_{ij}| = 0 \qquad [8.6]$$

Let (α_{ij}) be a sequence of scalars with $|\alpha_{ij}| \leq 1$ for all $(i,j) \in \mathbb{N} \times \mathbb{N}$. Then the result follows from [8.6] and the following inequality

$$|\alpha_{ij}x_{ij}| \leq |\alpha_{ij}||x_{ij}| \leq |x_{ij}| \ for\ all\ (i,j) \in \mathbb{N} \times \mathbb{N}.$$

That the space ${}_2\mathcal{Z}_0^I$ is monotone follows from the Lemma 1.16. For ${}_2m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 8.2.9. If I is not maximal, then the space ${}_2\mathcal{Z}^I$ is neither solid nor monotone.

Proof. Here we give a counter example. Let $(x_{ij}) = 1$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$. Then $(x_{ij}) \in {}_2\mathcal{Z}^I$. Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} \times \mathbb{N} - K \notin I$. Define the sequence

$$(y_{ij}) = \begin{cases} (x_{ij}), & \text{if } (i, j) \in K, \\ 0, & otherwise. \end{cases}$$

Then (y_{ij}) belongs to the canonical preimage of K-step space of ${}_2\mathcal{Z}^I$ but $(y_{ij}) \notin {}_2\mathcal{Z}^I$. Hence ${}_2\mathcal{Z}^I$ is not monotone.

Theorem 8.2.10. The spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are sequence algebras.

Proof. We prove that ${}_2\mathcal{Z}_0^I$ is a sequence algebra. Let $(x_{ij}), (y_{ij}) \in {}_2\mathcal{Z}_0^I$. Then

$$I - \lim |x_{ij}| = 0 \quad and \quad I - \lim |y_{ij}| = 0$$

Then we have $I - \lim |(x_{ij}.y_{ij})| = 0$. Thus $(x_{ij}.y_{ij}) \in {}_2\mathcal{Z}_0^I$. Hence ${}_2\mathcal{Z}_0^I$ is a sequence algebra. For the space ${}_2\mathcal{Z}^I$, the result can be proved similarly.

Theorem 8.2.11. The spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are not convergence free in general.

Proof. Here we give a counter example. Let $I = I_f$. Consider the sequence (x_{ij}) and (y_{ij}) defined by

$$x_{ij} = \frac{1}{i.j} \text{ and } y_{ij} = i.j \text{ for all (i,j)} \in \mathbb{N} \times \mathbb{N}$$

Then $(x_{ij}) \in {}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$, but $(y_{ij}) \notin {}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$. Hence the spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are not convergence free.

Theorem 8.2.12. If I is not maximal and $I \neq I_f$, then the spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are not symmetric.

Proof. Let $A \in I$ be infinite. If

$$x_{ij} = \begin{cases} 1, & \text{for } i\ ,\ j \in A, \\ 0, & otherwise. \end{cases}$$

Then $x_{ij} \in {}_2\mathcal{Z}_0^I \subset {}_2\mathcal{Z}^I$. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & otherwise. \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{(\pi(m)\pi(n))} \notin {}_2\mathcal{Z}^I$ and $x_{(\pi(m)\pi(n))} \notin {}_2\mathcal{Z}_0^I$. Hence ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are not symmetric.

Theorem 8.2.13. The sequence spaces ${}_2\mathcal{Z}^I$ and ${}_2\mathcal{Z}_0^I$ are linearly isomorphic to the spaces ${}_2c^I$ and ${}_2c_0^I$ respectively, i.e ${}_2\mathcal{Z}^I \cong {}_2c^I$ and ${}_2\mathcal{Z}_0^I \cong {}_2c_0^I$.

Proof. We shall prove the result for the space ${}_2\mathcal{Z}^I$ and ${}_2c^I$. The proof for the other spaces will follow similarly. We need to show that there exists a linear bijection between the spaces ${}_2\mathcal{Z}^I$ and ${}_2c^I$. Define a map $T :\ {}_2\mathcal{Z}^I \longrightarrow {}_2c^I$ such that $x \to x^{'} = Tx$

$$T(x_{ij}) = px_{ij} + (1-p)x_{(i-1)(j-1)} = x_{ij}^{'}$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$. Clearly T is linear. Further, it is trivial that $x = 0 = (0, 0, 0, ...)$ whenever $Tx = 0$ and hence injective. Let $x_{ij}^{'} \in {}_2c^I$ and define the sequence $x = x_{ij}$ by

$$x_{ij} = M \sum_{r=0}^{i} \sum_{s=0}^{j} (-1)^{(i-r)(j-s)} N^{(i-r)(j-s)} x_{ij}^{'}$$

for $(i,j) \in \mathbb{N} \times \mathbb{N}$ and where $M = \frac{1}{p}$ and $N = \frac{1-p}{p}$. Then we have

$$\lim_{(i,j)\to\infty} px_{ij} + (1-p)x_{(i-1)(j-1)} =$$

$$p \lim_{(i,j)\to\infty} M \sum_{r=0}^{i}\sum_{s=0}^{j}(-1)^{(i-r)(j-s)}N^{(i-r)(j-s)}x'_{ij}$$

$$+(1-p)\lim_{(i,j)\to\infty} M \sum_{r=0}^{i-1}\sum_{s=0}^{j-1}(-1)^{(i-1-r)(j-1-s)}N^{(i-1-r)(j-1-s)}x'_{(i-1)(j-1)}$$

$$= \lim_{(i,j)\to\infty} x'_{ij}$$

which shows that $x \in {}_2\mathcal{Z}^I$. Hence T is a linear bijection. Also we have $||x||_* = ||Z^p x||_c$. Therefore

$$||x||_* = \sup_{(i,j)\in\mathbb{N}\times\mathbb{N}} |px_{ij} + (1-p)x_{(i-1)(j-1)}|$$

$$= \sup_{(i,j)\in\mathbb{N}\times\mathbb{N}} |pM \sum_{r=0}^{i}\sum_{s=0}^{j}(-1)^{(i-r)(j-s)}N^{(i-r)(j-s)}x'_{ij}$$

$$+(1-p)M \sum_{r=0}^{i-1}\sum_{s=0}^{j-1}(-1)^{(i-1-r)(j-1-s)}N^{(i-1-r)(j-1-s)}x'_{(i-1)(j-1)}|$$

$$= \sup_{(i,j)\in\mathbb{N}\times\mathbb{N}} |x'_{ij}| = ||x'||_{\,{}_2c^I}.$$

Hence ${}_2\mathcal{Z}^I \cong {}_2c^I$.