

## **Chapter 6**

# **Zweier I-Convergent Sequence Spaces Defined by a Sequence of Modulii**



## 6.1 Introduction

Recently Khan and Ebadullah[31] introduced the following classes of sequences

$$\mathcal{Z}^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k - L|) \geq \varepsilon, \text{ for some } L \in \mathbb{C}\} \in I\},$$

$$\mathcal{Z}_0^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq \varepsilon\} \in I\},$$

$$\mathcal{Z}_\infty^I(f) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f(|x_k|) \geq M, \text{ for each fixed } M > 0\} \in I\}.$$

We also denote by

$$m_{\mathcal{Z}}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}^I(f)$$

and

$$m_{\mathcal{Z}_0}^I(f) = \mathcal{Z}_\infty^I(f) \cap \mathcal{Z}_0^I(f).$$

In this chapter we introduce the following class of sequence spaces.

$$\mathcal{Z}^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k - L|) \geq \varepsilon, \text{ for some } L \in \mathbb{C}\} \in I\},$$

$$\mathcal{Z}_0^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k|) \geq \varepsilon\} \in I\},$$

$$\mathcal{Z}_\infty^I(F) = \{(x_k) \in \omega : \{k \in \mathbb{N} : f_k(|x_k|) \geq M, \text{ for each fixed } M > 0\} \in I\}.$$

We also denote by

$$m_{\mathcal{Z}}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}^I(F)$$

and

$$m_{\mathcal{Z}_0}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}_0^I(F).$$

## 6.2 Main Results

**Theorem 6.2.1.** For a sequence of moduli  $F = (f_k)$ , the classes of sequences  $\mathcal{Z}^I(F)$ ,  $\mathcal{Z}_0^I(F)$ ,  $m_{\mathcal{Z}}^I(F)$  and  $m_{\mathcal{Z}_0}^I(F)$  are linear spaces.

**Proof.** We shall prove the result for the space  $\mathcal{Z}^I(F)$ . The proof for the other spaces will follow similarly. Let  $(x_k), (y_k) \in \mathcal{Z}^I(F)$  and let  $\alpha, \beta$  be scalars. Then

$$I - \lim f_k(|x_k - L_1|) = 0, \text{ for some } L_1 \in \mathbb{C} \quad ;$$

$$I - \lim f_k(|y_k - L_2|) = 0, \text{ for some } L_2 \in \mathbb{C} \quad ;$$

That is for a given  $\epsilon > 0$ , we have

$$A_1 = \{k \in \mathbb{N} : f_k(|x_k - L_1|) > \frac{\epsilon}{2}\} \in I, \quad [6.1]$$

$$A_2 = \{k \in \mathbb{N} : f_k(|y_k - L_2|) > \frac{\epsilon}{2}\} \in I. \quad [6.2]$$

Since  $f_k$  is a modulus function, we have

$$\begin{aligned} f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) &\leq f_k(|\alpha||x_k - L_1|) + f_k(|\beta||y_k - L_2|) \\ &\leq f_k(|x_k - L_1|) + f_k(|y_k - L_2|) \end{aligned}$$

Now, by [6.1] and [6.2],  $\{k \in \mathbb{N} : f_k(|(\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2)|) > \epsilon\} \subset A_1 \cup A_2$ . Therefore  $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I(F)$ . Hence  $\mathcal{Z}^I(F)$  is a linear space.

We state the following result without proof in view of Theorem 6.2.1.

**Theorem 6.2.2.** The spaces  $m_{\mathcal{Z}}^I(F)$  and  $m_{\mathcal{Z}_0}^I(F)$  are normed linear spaces, normed by

$$||x_k||_* = \sup_k f_k(|x_k|). \quad [6.3]$$

**Theorem 6.2.3.** A sequence  $x = (x_k) \in m_{\mathbb{Z}}^I(F)$  I-converges if and only if for every  $\epsilon > 0$  there exists  $N_\epsilon \in \mathbb{N}$  such that

$$\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m_{\mathbb{Z}}^I(F). \quad [6.4]$$

**Proof.** Suppose that  $L = I - \lim x$ . Then

$$B_\epsilon = \{k \in \mathbb{N} : |x_k - L| < \frac{\epsilon}{2}\} \in m_{\mathbb{Z}}^I(F). \text{ For all } \epsilon > 0.$$

Fix an  $N_\epsilon \in B_\epsilon$ . Then we have

$$|x_{N_\epsilon} - x_k| \leq |x_{N_\epsilon} - L| + |L - x_k| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all  $k \in B_\epsilon$ . Hence  $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m_{\mathbb{Z}}^I(F)$ .

Conversely, suppose that  $\{k \in \mathbb{N} : f_k(|x_k - x_{N_\epsilon}|) < \epsilon\} \in m_{\mathbb{Z}}^I(F)$ . That is  $\{k \in \mathbb{N} : (|x_k - x_{N_\epsilon}|) < \epsilon\} \in m_{\mathbb{Z}}^I(F)$  for all  $\epsilon > 0$ . Then the set

$$C_\epsilon = \{k \in \mathbb{N} : x_k \in [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]\} \in m_{\mathbb{Z}}^I(F) \text{ for all } \epsilon > 0.$$

Let  $J_\epsilon = [x_{N_\epsilon} - \epsilon, x_{N_\epsilon} + \epsilon]$ . If we fix an  $\epsilon > 0$  then we have  $C_\epsilon \in m_{\mathbb{Z}}^I(F)$  as well as  $C_{\frac{\epsilon}{2}} \in m_{\mathbb{Z}}^I(F)$ . Hence  $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m_{\mathbb{Z}}^I(F)$ . This implies that

$$J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{k \in \mathbb{N} : x_k \in J\} \in m_{\mathbb{Z}}^I(F)$$

that is

$$\text{diam} J \leq \text{diam} J_\epsilon$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq \dots \supseteq I_k \supseteq \dots$$

with the property that  $\text{diam} I_k \leq \frac{1}{2} \text{diam} I_{k-1}$  for  $(k=2,3,4,\dots)$  and  $\{k \in \mathbb{N} : x_k \in I_k\} \in m_{\mathbb{Z}}^I(F)$  for  $(k=1,2,3,4,\dots)$ . Then there exists a  $\xi \in \cap I_k$  where  $k \in \mathbb{N}$  such that  $\xi = I - \lim x$ . So that  $f_k(\xi) = I - \lim f_k(x)$ , that is  $L = I - \lim f_k(x)$ .

**Theorem 6.2.4.** Let  $(f_k)$  and  $(g_k)$  be modulus functions for some fixed  $k$  that satisfy the  $\Delta_2$ -condition. If  $X$  is any of the spaces  $\mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathbb{Z}}^I$  and  $m_{\mathbb{Z}_0}^I$  etc, then the following assertions hold.

- (a)  $X(g_k) \subseteq X(f_k \cdot g_k)$ ,
- (b)  $X(f_k) \cap X(g_k) \subseteq X(f_k + g_k)$ .

**Proof.** (a) Let  $(x_n) \in \mathcal{Z}_0^I(g_k)$ . Then

$$I - \lim_n g_k(|x_n|) = 0 \quad [6.5]$$

Let  $\epsilon > 0$  and choose  $\delta$  with  $0 < \delta < 1$  such that  $f_k(t) < \epsilon$  for  $0 < t < \delta$ . Write  $y_n = g_k(|x_n|)$  and consider  $\lim_n f_k(y_n) = \lim_n f_k(y_n)_{y_n < \delta} + \lim_n f_k(y_n)_{y_n > \delta}$ . We have

$$\lim_n f_k(y_n) \leq f_k(2) \lim_n (y_n). \quad [6.6]$$

For  $y_n > \delta$ , we have  $y_n < \frac{y_n}{\delta} < 1 + \frac{y_n}{\delta}$ . Since  $f_k$  is non-decreasing, it follows that

$$f_k(y_n) < f_k(1 + \frac{y_n}{\delta}) < \frac{1}{2} f_k(2) + \frac{1}{2} f_k(\frac{2y_n}{\delta})$$

Since  $f_k$  satisfies the  $\Delta_2$ -condition, we have

$$f_k(y_n) < \frac{1}{2} K \frac{y_n}{\delta} f_k(2) + \frac{1}{2} K \frac{y_n}{\delta} f_k(2) = K \frac{y_n}{\delta} f_k(2)$$

Hence

$$\lim_n f_k(y_n) \leq \max(1, K) \delta^{-1} f_k(2) \lim_n (y_n). \quad [6.7]$$

From [6.5], [6.6] and [6.7], we have  $(x_n) \in \mathcal{Z}_0^I(f_k \cdot g_k)$ .

Thus  $\mathcal{Z}_0^I(g_k) \subseteq \mathcal{Z}_0^I(f_k \cdot g_k)$ . The other cases can be proved similarly.

(b) Let  $(x_n) \in \mathcal{Z}_0^I(f_k) \cap \mathcal{Z}_0^I(g_k)$ . Then

$$I - \lim_n f_k(|x_n|) = 0 \text{ and } I - \lim_n g_k(|x_n|) = 0$$

The rest of the proof follows from the following equality

$$\lim_n (f_k + g_k)(|x_n|) = \lim_n f_k(|x_n|) + \lim_n g_k(|x_n|).$$

Corollary 6.2.5.  $X \subseteq X(f_k)$  for some fixed  $k$  and  $X = \mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$  and  $m_{\mathcal{Z}_0}^I$ .

Theorem 6.2.6. The spaces  $\mathcal{Z}_0^I(F)$  and  $m_{\mathcal{Z}_0}^I(F)$  are solid and monotone.

Proof. We shall prove the result for  $\mathcal{Z}_0^I(F)$ . Let  $(x_k) \in \mathcal{Z}_0^I(F)$ . Then

$$I - \lim_k f_k(|x_k|) = 0 \quad [6.8]$$

Let  $(\alpha_k)$  be a sequence of scalars with  $|\alpha_k| \leq 1$  for all  $k \in \mathbb{N}$ . Then the result follows from [6.8] and the following inequality

$$f_k(|\alpha_k x_k|) \leq |\alpha_k| f_k(|x_k|) \leq f_k(|x_k|) \text{ for all } k \in \mathbb{N}.$$

That the space  $\mathcal{Z}_0^I(F)$  is monotone follows from the Lemma 6.1.1. For  $m_{\mathcal{Z}_0}^I(F)$  the result can be proved similarly.

Theorem 6.2.7. The spaces  $\mathcal{Z}^I(F)$  and  $m_{\mathcal{Z}}^I(F)$  are neither solid nor monotone in general.

Proof. Here we give a counter example. Let  $I = I_\delta$  and  $f_k(x) = x^2$  for some fixed  $k$  and for all  $x \in [0, \infty)$ . Consider the K-step space  $X_K(f_k)$  of  $X$  defined as follows.

Let  $(x_n) \in X$  and let  $(y_n) \in X_K$  be such that

$$(y_n) = \begin{cases} (x_n), & \text{if } n \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Consider the sequence  $(x_n)$  defined by  $(x_n) = 1$  for all  $n \in \mathbb{N}$ . Then  $(x_n) \in \mathcal{Z}^I(F)$  but its  $K$ -stepspace preimage does not belong to  $\mathcal{Z}^I(F)$ . Thus  $\mathcal{Z}^I(F)$  is not monotone. Hence  $\mathcal{Z}^I(F)$  is not solid.

Theorem 6.2.8. The spaces  $\mathcal{Z}^I(F)$  and  $\mathcal{Z}_0^I(F)$  are sequence algebras.

Proof. We prove that  $\mathcal{Z}_0^I(F)$  is a sequence algebra. Let  $(x_k), (y_k) \in \mathcal{Z}_0^I(F)$ . Then

$$I - \lim f_k(|x_k|) = 0$$

and

$$I - \lim f_k(|y_k|) = 0$$

Then we have

$$I - \lim f_k(|(x_k \cdot y_k)|) = 0$$

Thus  $(x_k \cdot y_k) \in \mathcal{Z}_0^I(F)$  is a sequence algebra. For the space  $\mathcal{Z}^I(F)$ , the result can be proved similarly.

Theorem 6.2.9. The spaces  $\mathcal{Z}^I(F)$  and  $\mathcal{Z}_0^I(F)$  are not convergence free in general.

Proof. Here we give a counter example. Let  $I = I_f$  and  $f_k(x) = x^3$  for some fixed  $k$  and for all  $x \in [0, \infty)$ . Consider the sequence  $(x_n)$  and  $(y_n)$  defined by

$$x_n = \frac{1}{n} \quad \text{and} \quad y_n = n \quad \text{for all } n \in \mathbb{N}$$



Then  $(x_n) \in \mathcal{Z}^I(F)$  and  $\mathcal{Z}_0^I(F)$ , but  $(y_n) \notin \mathcal{Z}^I(F)$  and  $\mathcal{Z}_0^I(F)$ . Hence the spaces  $\mathcal{Z}_0^I(F)$  and  $\mathcal{Z}^I(F)$  are not convergence free.

Theorem 6.2.10. If  $I$  is not maximal and  $I \neq I_f$ , then the spaces  $\mathcal{Z}^I(F)$  and  $\mathcal{Z}_0^I(F)$  are not symmetric.

Proof. Let  $A \in I$  be infinite and  $f_k(x) = x$  for some fixed  $k$  and for all  $x \in [0, \infty)$ .

If

$$x_n = \begin{cases} 1, & \text{for } n \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Then by lemma 1.22  $(x_n) \in \mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F)$ . Let  $K \subset \mathbb{N}$  be such that  $K \notin I$  and  $\mathbb{N} - K \notin I$ . Let  $\phi : K \rightarrow A$  and  $\psi : \mathbb{N} - K \rightarrow \mathbb{N} - A$  be bijections, then the map  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  defined by

$$\pi(n) = \begin{cases} \phi(n), & \text{for } n \in K, \\ \psi(n), & \text{otherwise.} \end{cases}$$

is a permutation on  $\mathbb{N}$ , but  $x_{\pi(n)} \notin \mathcal{Z}^I(F)$  and  $x_{\pi(n)} \notin \mathcal{Z}_0^I(F)$ . Hence  $\mathcal{Z}^I(F)$  and  $\mathcal{Z}_0^I(F)$  are not symmetric.

Theorem 6.2.11.  $\mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F) \subset \mathcal{Z}_\infty^I(F)$ .

Proof. Let  $(x_k) \in \mathcal{Z}^I(F)$ . Then there exists  $L \in \mathbb{C}$  such that

$$I - \lim f_k(|x_k - L|) = 0$$

We have  $f_k(|x_k|) \leq \frac{1}{2}f_k(|x_k - L|) + f_k(\frac{1}{2}(|L|))$ . Taking the supremum over  $k$  on both sides we get  $(x_k) \in \mathcal{Z}_\infty^I(F)$ . The inclusion  $\mathcal{Z}_0^I(F) \subset \mathcal{Z}^I(F)$  is obvious.

Theorem 6.2.12. The function  $\hbar : m_{\mathcal{Z}}^I(F) \rightarrow \mathbb{R}$  is the Lipschitz function, where  $m_{\mathcal{Z}}^I(F) = \mathcal{Z}_\infty^I(F) \cap \mathcal{Z}^I(F)$ , and hence uniformly continuous.

**Proof.** Let  $x, y \in m_{\mathcal{Z}}^I(F)$ ,  $x \neq y$ . Then the sets

$$A_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| \geq \|x - y\|_*\} \in I,$$

$$A_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| \geq \|x - y\|_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \|x - y\|_*\} \in m_{\mathcal{Z}}^I(F),$$

$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \|x - y\|_*\} \in m_{\mathcal{Z}}^I(F).$$

Hence also  $B = B_x \cap B_y \in m_{\mathcal{Z}}^I(F)$ , so that  $B \neq \phi$ . Now taking  $k$  in  $B$ ,

$$|\hbar(x) - \hbar(y)| \leq |\hbar(x) - x_k| + |x_k - y_k| + |y_k - \hbar(y)| \leq 3\|x - y\|_*.$$

Thus  $\hbar$  is a Lipschitz function. For the space  $m_{\mathcal{Z}_0}^I(F)$  the result can be proved similarly.

**Theorem 6.2.13.** If  $x, y \in m_{\mathcal{Z}}^I(F)$ , then  $(x, y) \in m_{\mathcal{Z}}^I(F)$  and  $\hbar(xy) = \hbar(x)\hbar(y)$ .

**Proof.** For  $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x_k - \hbar(x)| < \epsilon\} \in m_{\mathcal{Z}}^I(F),$$

$$B_y = \{k \in \mathbb{N} : |y_k - \hbar(y)| < \epsilon\} \in m_{\mathcal{Z}}^I(F).$$

Now,

$$\begin{aligned} |x_k y_k - \hbar(x)\hbar(y)| &= |x_k y_k - x_k \hbar(y) + x_k \hbar(y) - \hbar(x)\hbar(y)| \\ &\leq |x_k| |y_k - \hbar(y)| + |\hbar(y)| |x_k - \hbar(x)| \end{aligned} \quad [6.9]$$

As  $m_{\mathcal{Z}}^I(F) \subseteq \mathcal{Z}_{\infty}^I(F)$ , there exists an  $M \in \mathbb{R}$  such that  $|x_k| < M$  and  $|\hbar(y)| < M$ .

Using eqn[6.9] we get

$$|x_k y_k - \hbar(x)\hbar(y)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all  $k \in B_x \cap B_y \in m^I(F)$ . Hence  $(x.y) \in m_{\mathbb{Z}}^I(F)$  and  $\hbar(xy) = \hbar(x)\hbar(y)$ . For the space  $m_{\mathbb{Z}_0}^I(F)$  the result can be proved similarly.

