Chapter 2

Zweier I-Convergent Sequence Spaces

"In most sciences one generation tears down what another has built and what one has established another undoes. In mathematics alone each generation builds a new story to the old structure."- Hankel.

2.1 Introduction

Let l_∞, c and c_0 denote the Banach spaces of bounded,convergent and null sequences respectively normed by $||x||_\infty = \sup_k |x_k|$.

Each linear subspace of ω, for example, $\lambda, \mu \subset \omega$ is called a sequence space.

A sequence space X with linear topology is called a K-space provided each of maps $p_i : X \longrightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$.

A K-space λ is called an FK-space provided λ is a complete linear metric space.

An FK-space whose topology is normable is called a BK-space.

Let λ and μ be two sequence spaces and $A = (a_{nk})$ is an infinite matrix of real or complex numbers (a_{nk}), where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from λ to μ, and we denote it by writing
$A : \lambda \longrightarrow \mu$.

If for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A transform of x is in μ, where

$$(Ax)_n = \sum_k a_{nk}x_k, \qquad (n \in \mathbb{N}). \tag{2.1}$$

By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \longrightarrow \mu$. Thus, $A \in (\lambda : \mu)$ if and only if series on the right side of [2.1] converges for each $n \in \mathbb{N}$ and every $x \in \lambda$.

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay,Başar and Mursaleen[1], Başar and Altay[3], Malkowsky[57], Ng and Lee[59], and Wang[74]. Şengönül[68] defined the sequence $y = (y_i)$ which is frequently used as the Z^p transform of the sequence $x = (x_i)$ i.e,

$$y_i = px_i + (1-p)x_{i-1}$$

where $x_{-1} = 0, p \neq 1$, $1 < p < \infty$ and Z^p denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, & (i = k), \\ 1-p, & (i-1 = k); (i, k \in \mathbb{N}), \\ 0, & \text{otherwise.} \end{cases}$$

Following Başar and Altay[3], Şengönül[68] introduced the Zweier sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ as follows

$$\mathcal{Z} = \{x = (x_k) \in \omega : Z^p x \in c\}$$

$$\mathcal{Z}_0 = \{x = (x_k) \in \omega : Z^p x \in c_0\}.$$

Here we list below some of the results of [68] which we will need as a reference in order to establish analogously some of the results of this article.

Theorem 2.1.1. [68, Theorem 2.1] The sets $\mathcal{Z}$ and $\mathcal{Z}_0$ are the linear spaces with the co-ordinate wise addition and scalar multiplication which are the BK-spaces with the norm

$$||x||_{\mathcal{Z}} = ||x||_{\mathcal{Z}_0} = ||Z^p x||_c.$$

Theorem 2.1.2. [68, Theorem 2.2] The sequence spaces $\mathcal{Z}$ and $\mathcal{Z}_0$ are linearly isomorphic to the spaces c and c_0 respectively, i.e $\mathcal{Z} \cong c$ and $\mathcal{Z}_0 \cong c_0$[See (Theorem 2.2.[18])]

Theorem 2.1.3. [68, Theorem 2.3] The inclusions $\mathcal{Z}_0 \subset \mathcal{Z}$ strictly hold for $p \neq 1$.

Theorem 2.1.4. [68, Theorem 2.6] $\mathcal{Z}_0$ is solid.

Theorem 2.1.5. [68, Theorem 3.6] $\mathcal{Z}$ is not a solid sequence space.

The following Lemmas will be used for establishing some results of this article.

Lemma 2.1.6. Let E be a sequence space. If E is solid then E is monotone. (see [20], page 53).

Lemma 2.1.7. If I $\subset 2^{\mathbb{N}}$ and M$\subseteq \mathbb{N}$. If M $\notin$I, then M$\cap\mathbb{N} \notin$I.
(see [71-72]).

2.2 Main Results

In this chapter we introduce the following classes of sequence spaces.

$$\mathcal{Z}^I = \{x = (x_k) \in \omega : \{k \in \mathbb{N} : I - \lim Z^p x = L, \text{ for some L} \in \mathbb{C}\} \in I\}$$

$$\mathcal{Z}_0^I = \{x = (x_k) \in \omega : \{k \in \mathbb{N} : I - \lim Z^p x = 0\} \in I\}$$

$$\mathcal{Z}_\infty^I = \{x = (x_k) \in \omega : \sup_k |Z^p x| < \infty\}.$$

We also denote by

$$m_{\mathcal{Z}}^I = \mathcal{Z}_\infty \cap \mathcal{Z}^I$$

and

$$m_{\mathcal{Z}_0}^I = \mathcal{Z}_\infty \cap \mathcal{Z}_0^I$$

Throughout the article, for the sake of convenience now we will denote by $Z^p(x_k) = x^/, Z^p(y_k) = y^/, Z^p(z_k) = z^/$ for $x, y, z \in \omega$.

Theorem 2.2.1. The classes of sequences $\mathcal{Z}^I, \mathcal{Z}_0^I, m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$ are linear spaces.

Proof. We shall prove the result for the space $\mathcal{Z}^I$. The proof for the other spaces will follow similarly. Let $(x_k), (y_k) \in \mathcal{Z}^I$ and let α, β be scalars. Then

$$I - \lim |x_k^{/} - L_1| = 0, \text{ for some } L_1 \in \mathbb{C};$$

$$I - \lim |y_k^{/} - L_2| = 0, \text{ for some } L_2 \in \mathbb{C};$$

That is for a given $\epsilon > 0$, we have

$$A_1 = \{k \in \mathbb{N} : |x_k^{/} - L_1| > \frac{\epsilon}{2}\} \in I, \tag{2.2}$$

$$A_2 = \{k \in \mathbb{N} : |y_k^{/} - L_2| > \frac{\epsilon}{2}\} \in I. \tag{2.3}$$

we have

$$\begin{aligned} |(\alpha x_k^{/} + \beta y_k^{/}) - (\alpha L_1 + \beta L_2)| &\leq |\alpha|(|x_k^{/} - L_1|) + |\beta|(|y_k^{/} - L_2|) \\ &\leq |x_k^{/} - L_1| + |y_k^{/} - L_2| \end{aligned}$$

Now, by [2.2] and [2.3], $\{k \in \mathbb{N} : |(\alpha x_k^{/} + \beta y_k^{/}) - (\alpha L_1 + \beta L_2)| > \epsilon\} \subset A_1 \cup A_2$. Therefore $(\alpha x_k + \beta y_k) \in \mathcal{Z}^I$

Hence $\mathcal{Z}^I$ is a linear space.

Theorem 2.2.2. The spaces $m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$ are normed linear spaces,normed by

$$||x_k^{/}||_* = \sup_k |Z^p(x)|. \tag{2.4}$$

where $x_k^{/} = Z^p(x)$

Proof. It is clear from Theorem 2.2.1 that $m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$ are linear spaces. It is easy to verify that [2.4] defines a norm on the spaces $m_{\mathcal{Z}}^I$ and $m_{\mathcal{Z}_0}^I$.

Theorem 2.2.3. A sequence $x = (x_k) \in m_{\mathcal{Z}}^I$ I-converges if and only if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that

$$\{k \in \mathbb{N} : |x_k^/ - x_{N_\epsilon}^/| < \epsilon\} \in m_{\mathcal{Z}}^I \quad [2.5]$$

Proof. Suppose that $L = I - \lim x^/$. Then

$$B_\epsilon = \{k \in \mathbb{N} : |x_k^/ - L| < \frac{\epsilon}{2}\} \in m_{\mathcal{Z}}^I \text{ for all } \epsilon > 0.$$

Fix an $N_\epsilon \in B_\epsilon$. Then we have

$$|x_{N_\epsilon}^/ - x_k^/| \leq |x_{N_\epsilon}^/ - L| + |L - x_k^/| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which holds for all $k \in B_\epsilon$.

Hence $\{k \in \mathbb{N} : |x_k^/ - x_{N_\epsilon}^/| < \epsilon\} \in m_{\mathcal{Z}}^I$.

Conversely, suppose that $\{k \in \mathbb{N} : |x_k^/ - x_{N_\epsilon}^/| < \epsilon\} \in m_{\mathcal{Z}}^I$. That is $\{k \in \mathbb{N} : |x_k^/ - x_{N_\epsilon}^/| < \epsilon\} \in m_{\mathcal{Z}}^I$ for all $\epsilon > 0$. Then the set

$$C_\epsilon = \{k \in \mathbb{N} : x_k^/ \in [x_{N_\epsilon}^/ - \epsilon, x_{N_\epsilon}^/ + \epsilon]\} \in m_{\mathcal{Z}}^I \text{ for all } \epsilon > 0.$$

Let $J_\epsilon = [x_{N_\epsilon}^/ - \epsilon, x_{N_\epsilon}^/ + \epsilon]$. If we fix an $\epsilon > 0$ then we have $C_\epsilon \in m_{\mathcal{Z}}^I$ as well as $C_{\frac{\epsilon}{2}} \in m_{\mathcal{Z}}^I$. Hence $C_\epsilon \cap C_{\frac{\epsilon}{2}} \in m_{\mathcal{Z}}^I$. This implies that

$$J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$$

that is

$$\{k \in \mathbb{N} : x_k^/ \in J\} \in m_{\mathcal{Z}}^I$$

that is

$$diam J \leq diam J_\epsilon$$

where the diam of J denotes the length of interval J. In this way, by induction we get the sequence of closed intervals

$$J_\epsilon = I_0 \supseteq I_1 \supseteq ..... \supseteq I_k \supseteq ....$$

with the property that $diamI_k \leq \frac{1}{2}diamI_{k-1}$ for (k=2,3,4,.....) and $\{k \in \mathbb{N} : x_k^{/} \in I_k\} \in m_{\mathcal{Z}}^I$ for (k=1,2,3,4,......). Then there exists a $\xi \in \cap I_k$ where $k \in \mathbb{N}$ such that $\xi^{/} = I - \lim x^{/}$, that is $L = I - \lim x^{/}$.

Theorem 2.2.4. Let I be an admissible ideal. Then the following are equivalent.

(a) $(x_k) \in \mathcal{Z}^I$;

(b) there exists $(y_k) \in \mathcal{Z}$ such that $x_k = y_k$, for a.a.k.r.I;

(c) there exists $(y_k) \in \mathcal{Z}$ and $(z_k) \in \mathcal{Z}_0^I$ such that $x_k = y_k + z_k$ for all $k \in \mathbb{N}$ and $\{k \in \mathbb{N} : |y_k - L| \geq \epsilon\} \in I$;

(d) there exists a subset $K = \{k_1 < k_2....\}$ of $\mathbb{N}$ such that $K \in \pounds(I)$ and $\lim\limits_{n\to\infty} |x_{k_n} - L| = 0$.

Proof. (a) implies (b). Let $(x_k) \in \mathcal{Z}^I$. Then there exists $L \in \mathbb{C}$ such that

$$\{k \in \mathbb{N} : |x_k^{/} - L| \geq \epsilon\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{k \leq m_t : |x_k^{/} - L| \geq \frac{1}{t}\} \in I.$$

Define a sequence (y_k) as

$$y_k = x_k, \text{ for all } k \leq m_1.$$

For $m_t < k \leq m_{t+1}, t \in \mathbb{N}$.

$$y_k = \begin{cases} x_k, & \text{if } |x_k^{/} - L| < t^{-1}, \\ L, & \text{otherwise.} \end{cases}$$

Then $(y_k) \in \mathcal{Z}$ and form the following inclusion

$$\{k \leq m_t : x_k \neq y_k\} \subseteq \{k \leq m_t : |x_k^{/} - L| \geq \epsilon\} \in I.$$

We get $x_k = y_k$, for a.a.k.r.I.

(b) implies (c). For $(x_k) \in \mathcal{Z}^I$. Then there exists $(y_k) \in \mathcal{Z}$ such that $x_k = y_k$, for a.a.k.r.I. Let $K = \{k \in \mathbb{N} : x_k \neq y_k\}$, then $K \in I$. Define a sequence (z_k) as

$$z_k = \begin{cases} x_k - y_k, & \text{if } k \in K, \\ 0, & \text{otherwise.} \end{cases}$$

Then $z_k \in \mathcal{Z}_0^I$ and $y_k \in \mathcal{Z}$.

(c) implies (d). Let $P_1 = \{k \in \mathbb{N} : |z_k| \geq \epsilon\} \in I$ and

$$K = P_1^c = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I).$$

Then we have $\lim\limits_{n\to\infty} |x_{k_n} - L| = 0$.

(d) implies (a). Let $K = \{k_1 < k_2 < k_3 < ...\} \in \pounds(I)$ and $\lim\limits_{n\to\infty} |x_{k_n} - L| = 0$. Then for any $\epsilon > 0$, and Lemma , we have

$$\{k \in \mathbb{N} : |x_k^{/} - L| \geq \epsilon\} \subseteq K^c \cup \{k \in K : |x_k^{/} - L| \geq \epsilon\}.$$

Thus $(x_k) \in \mathcal{Z}^I$.

Theorem 2.2.5. The inclusions $\mathcal{Z}_0^I \subset \mathcal{Z}^I \subset \mathcal{Z}_\infty^I$ are proper.

Proof. Let $(x_k) \in \mathcal{Z}^I$. Then there exists L $\in \mathbb{C}$ such that

$$I - \lim |x_k^{/} - L| = 0$$

We have $|x_k^{/}| \leq \frac{1}{2}|x_k^{/} - L| + \frac{1}{2}|L|$. Taking the supremum over k on both sides we get $(x_k) \in \mathcal{Z}_\infty^I$. The inclusion $\mathcal{Z}_0^I \subset \mathcal{Z}^I$ is obvious.

Theorem 2.2.6. The function $\hbar : m^I_{\mathcal{Z}} \to \mathbb{R}$ is the Lipschitz function, where
$m^I_{\mathcal{Z}} = \mathcal{Z}^I \cap \mathcal{Z}_\infty$, and hence uniformly continuous.

Proof. Let $x, y \in m^I_{\mathcal{Z}}$, $x \neq y$. Then the sets

$$A_x = \{k \in \mathbb{N} : |x^/_k - \hbar(x^/)| \geq ||x^/ - y^/||_*\} \in I,$$

$$A_y = \{k \in \mathbb{N} : |y^/_k - \hbar(y^/)| \geq ||x^/ - y^/||_*\} \in I.$$

Thus the sets,

$$B_x = \{k \in \mathbb{N} : |x^/_k - \hbar(x^/)| < ||x^/ - y^/||_*\} \in m^I_{\mathcal{Z}},$$

$$B_y = \{k \in \mathbb{N} : |y^/_k - \hbar(y^/)| < ||x^/ - y^/||_*\} \in m^I_{\mathcal{Z}}.$$

Hence also $B = B_x \cap B_y \in m^I_{\mathcal{Z}}$, so that $B \neq \phi$. Now taking k in B,

$$|\hbar(x^/) - \hbar(y^/)| \leq |\hbar(x^/) - x^/_k| + |x^/_k - y^/_k| + |y^/ - \hbar(y^/)| \leq 3||x^/ - y^/||_*.$$

Thus $\hbar$ is a Lipschitz function. For $m^I_{\mathcal{Z}_0}$ the result can be proved similarly.

Theorem 2.2.7. If $x, y \in m^I_{\mathcal{Z}}$, then $(x.y) \in m^I_{\mathcal{Z}}$ and $\hbar(xy) = \hbar(x)\hbar(y)$.

Proof. For $\epsilon > 0$

$$B_x = \{k \in \mathbb{N} : |x^/ - \hbar(x^/)| < \epsilon\} \in m^I_{\mathcal{Z}},$$

$$B_y = \{k \in \mathbb{N} : |y^/ - \hbar(y^/)| < \epsilon\} \in m^I_{\mathcal{Z}}.$$

Now,

$$\begin{aligned}|x^/.y^/ - \hbar(x^/)\hbar(y^/)| &= |x^/.y^/ - x^/\hbar(y^/) + x^/\hbar(y^/) - \hbar(x^/)\hbar(y^/)| \\ &\leq |x^/||y^/ - \hbar(y^/)| + |\hbar(y^/)||x^/ - \hbar(x^/)| \qquad [2.6]\end{aligned}$$

As $m_{\mathcal{Z}}^I \subseteq \mathcal{Z}_\infty$, there exists an $M \in \mathbb{R}$ such that $|x^/| < M$ and $|\hbar(y^/)| < M$. Using eqn[2.6] we get

$$|x^/.y^/ - \hbar(x^/)\hbar(y^/)| \leq M\epsilon + M\epsilon = 2M\epsilon$$

For all $k \in B_x \cap B_y \in m_{\mathcal{Z}}^I$. Hence $(x.y) \in m_{\mathcal{Z}}^I$ and $\hbar(xy) = \hbar(x)\hbar(y)$. For $m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 2.2.8. The spaces $\mathcal{Z}_0^I$ and $m_{\mathcal{Z}_0}^I$ are solid and monotone .

Proof. We shall prove the result for $\mathcal{Z}_0^I$. Let $(x_k) \in \mathcal{Z}_0^I$. Then

$$I - \lim_k |x_k^/| = 0 \qquad [2.7]$$

Let (α_k) be a sequence of scalars with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then the result follows from [2.7] and the following inequality $|\alpha_k x_k^/| \leq |\alpha_k||x_k^/| \leq |x_k^/|$ for all $k \in \mathbb{N}$. That the space $\mathcal{Z}_0^I$ is monotone follows from the Lemma 2.1.6. For $m_{\mathcal{Z}_0}^I$ the result can be proved similarly.

Theorem 2.2.9. The spaces $\mathcal{Z}^I$ and $m_{\mathcal{Z}}^I$ are neither monotone nor solid, if I is neither maximal nor $I = I_f$ in general .

Proof. Here we give a counter example. Let I = I_δ. Consider the K-step space X_K of X defined as follows, Let $(x_k) \in X$ and let $(y_k) \in X_K$ be such that

$$(y_k^/) = \begin{cases} (x_k^/), & \text{if k is odd,} \\ 1, & otherwise. \end{cases}$$

Consider the sequence $(x_k^/)$ defined by $(x_k^/) = k^{-1}$ for all $k \in \mathbb{N}$. Then $(x_k) \in \mathcal{Z}^I$ but its K-stepspace preimage does not belong to $\mathcal{Z}^I$. Thus $\mathcal{Z}^I$ is not monotone. Hence $\mathcal{Z}^I$ is not solid.

Theorem 2.2.10. The spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are sequence algebras.

Proof. We prove that $\mathcal{Z}_0^I$ is a sequence algebra. Let $(x_k), (y_k) \in \mathcal{Z}_0^I$. Then

$$I - \lim |x_k^{/}| = 0$$

and

$$I - \lim |y_k^{/}| = 0$$

Then we have

$$I - \lim |(x_k^{/}.y_k^{/})| = 0$$

Thus $(x_k.y_k) \in \mathcal{Z}_0^I$. Hence $\mathcal{Z}_0^I$ is a sequence algebra. For the space $\mathcal{Z}^I$, the result can be proved similarly.

Theorem 2.2.11. The spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not convergence free in general.

Proof. Here we give a counter example. Let $I = I_f$. Consider the sequence $(x_k^{/})$ and $(y_k^{/})$ defined by

$$x_k^{/} = \frac{1}{k} \text{ and } y_k^{/} = k \text{ for all } k \in \mathbb{N}$$

Then $(x_k) \in \mathcal{Z}^I$ and $\mathcal{Z}_0^I$, but $(y_k) \notin \mathcal{Z}^I$ and $\mathcal{Z}_0^I$. Hence the spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not convergence free.

Theorem 2.2.12. If I is not maximal and $I \neq I_f$, then the spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not symmetric.

Proof. Let $A \in I$ be infinite. If

$$x_k^{/} = \begin{cases} 1, & \text{for } k \in A, \\ 0, & otherwise. \end{cases}$$

Then by lemma 1.16. $x_k \in \mathcal{Z}_0^I \subset \mathcal{Z}^I$. Let $K \subset \mathbb{N}$ be such that $K \notin I$ and $\mathbb{N} - K \notin I$. Let $\phi : K \to A$ and $\psi : \mathbb{N} - K \to \mathbb{N} - A$ be bijections, then the map $\pi : \mathbb{N} \to \mathbb{N}$ defined by

$$\pi(k) = \begin{cases} \phi(k), & \text{for } k \in K, \\ \psi(k), & otherwise. \end{cases}$$

is a permutation on $\mathbb{N}$, but $x_{\pi(k)} \notin \mathcal{Z}^I$ and $x_{\pi(k)} \notin \mathcal{Z}_0^I$. Hence $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are not symmetric.

Theorem 2.2.13. The sequence spaces $\mathcal{Z}^I$ and $\mathcal{Z}_0^I$ are linearly isomorphic to the spaces c^I and c_0^I respectively, i.e $\mathcal{Z}^I \cong c^I$ and $\mathcal{Z}_0^I \cong c_0^I$.

Proof. We shall prove the result for the space $\mathcal{Z}^I$ and c^I. The proof for the other spaces will follow similarly. We need to show that there exists a linear bijection between the spaces $\mathcal{Z}^I$ and c^I. Define a map $T : \mathcal{Z}^I \longrightarrow c^I$ such that $x \to x^/ = Tx$

$$T(x_k) = px_k + (1-p)x_{k-1} = x_k^/$$

where $x_{-1} = 0, p \neq 1, 1 < p < \infty$. Clearly T is linear. Further, it is trivial that $x = 0 = (0, 0, 0, ...)$ whenever $Tx = 0$ and hence injective. Let $x_k^/ \in c^I$ and define the sequence $x = x_k$ by

$$x_k = M \sum_{i=0}^{k} (-1)^{k-i} N^{k-i} x_i^/. \quad (i \in \mathbb{N})$$

where $M = \frac{1}{p}$ and $N = \frac{1-p}{p}$. Then we have

$$\lim_{k\to\infty} px_k + (1-p)x_{k-1} = p \lim_{k\to\infty} M \sum_{i=0}^{k} (-1)^{k-i} N^{k-i} x_i^/ +$$

$$(1-p) \lim_{k\to\infty} M \sum_{i=0}^{k-1} (-1)^{k-i} N^{k-i} x_i^/ = \lim_{k\to\infty} x_k^/$$

which shows that $x \in \mathcal{Z}^I$.

Hence T is a linear bijection. Also we have $||x||_* = ||Z^p x||_c$. Therefore

$$||x||_* = \sup_{k\in\mathbb{N}} |px_k + (1-p)x_{k-1}|$$

$$= \sup_{k\in\mathbb{N}} |pM\sum_{i=0}^{k}(-1)^{k-i}N^{k-i}x_i^{/} + (1-p)M\sum_{i=0}^{k-1}(-1)^{k-i}N^{k-i}x_i^{/}|$$

$$= \sup_{k\in\mathbb{N}} |x_k^{/}| = ||x^{/}||_{c^I}$$

Hence $\mathcal{Z}^I \cong c^I$.