Chapter 7

The Universe

In this chapter homogeneous, isotropic cosmological models are studied. The differential equations which describe these models together with the solutions are stated. Flat space-time theory of gravitation implies non-singular cosmological solutions, i.e. a big bang does not exist. This chapter follows along the lines of the articles $|Pet\ 90a|$ and $[Pet\ 90b]$.

7.1 Homogeneous Isotropic Cosmological Models with Cosmological Constant

In this chapter homogeneous isotropic cosmological models are studied. The field equations are given and the solutions are derived. There is no big bang.

We tart from the flat space-time theory of gravitation stated in chapter I. We use the flat space-time metric (1.1) with the pseudo-Euclidean geometry (1.5). The tensors of matter $T(M)_j^i$, of radiation $T(R)_j^i$ and of the cosmological constant $T(\Lambda)_j^i$ are given by (1.28) with

$$p = 0 \quad \text{(dust)} \tag{7.1a}$$

with

$$p = \frac{1}{3}\rho \quad \text{(radiation)} \tag{7.1b}$$

and by (1.21b) with (1.10). The energy-momentum tensor of gravitation is stated by relation (1.21a).

In the following, it is assumed that the universe is homogeneous and isotropic and matter is described in the rest frame i.e.

$$u^i = 0 \quad \text{(i=1,2,3).} \tag{7.2}$$

Then, the potentials are described by two time-dependent functions $a(t)$ and $h(t)$ with

$$\begin{aligned} g_{ij} &= a^2(t), (i = j = 1,2,3) \\ &= -1/h(t), (i = j = 4) \\ &= 0. (i \neq j) \end{aligned} \tag{7.3}$$

Then, the differential equations (1.24) with (1.23) yield

$$\left(a^3\sqrt{h}\frac{a'}{a}\right)' = 2\kappa c^4\left(\frac{1}{2}\rho_m + \frac{1}{3}\rho_r + \frac{\Lambda}{2\kappa c^2}\frac{a^3}{\sqrt{h}}\right) \tag{7.4a}$$

$$\left(a^3\sqrt{h}\frac{h'}{h}\right)' = 4\kappa c^4\left(\frac{1}{2}\rho_m + \rho_r - \frac{\Lambda}{2\kappa c^2}\frac{a^3}{\sqrt{h}} + \frac{1}{8\kappa c^2}L_G\right) \tag{7.4b}$$

where

$$L_G = \frac{1}{c^2}a^3\sqrt{h}\left(12\left(\frac{a\prime}{a}\right)^2 + \left(\frac{h\prime}{h}\right)^2 - \frac{1}{2}\left(-6\frac{a'}{a} + \frac{h\prime}{h}\right)^2\right) \tag{7.4c}$$

Here, ρ_m and ρ_r are the densities of matter and radiation, where the relations (7.1) are used and the prime denotes the t-derivative. It follows from (1.12) by the use of relation (7.2)

$$(u^i) = \left(\frac{dx^i}{d\tau}\right) = \left(0,0,0,c\sqrt{h}\right). \tag{7.5}$$

The proper-time is given by (1.8) implying

$$(cd\tau)^2 = -a^2[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + \frac{1}{h}(dct)^2. \tag{7.6}$$

The tensor of matter plus radiation has the form

$$\begin{aligned} T(M)^i_j + T(R)^i_j &= \frac{1}{3}\rho_r c^2 (i = j = 1,2,3) \\ &= -(\rho_m + \rho_r)c^2 (i = j = 4) \\ &= 0 (i \neq j) \end{aligned} \tag{7.7a}$$

and the tensor of gravitation is

$$\begin{aligned} T(G)^i_j &= \frac{1}{16\kappa}L_G \qquad (i=j=1, 2, 3) \\ &= -\frac{1}{16\kappa}L_G \qquad (i=j=4) \\ &= 0. \qquad (i\neq j) \end{aligned} \tag{7.7b}$$

The conservation law of the whole energy-momentum (1.25a) yields with i=4 by the use of (1.23c)

$$(\rho_m + \rho_r)c^2 + \frac{1}{16\kappa}L_G + \frac{\Lambda}{2\kappa}\frac{a^3}{\sqrt{h}} = \lambda c^2 \tag{7.8}$$

where λ is a constant of integration. It follows from the equations of motion (1.26)

$$\frac{d}{dt}(\rho_m + \rho_r) = -3\frac{a'}{a}p_r - \frac{1}{2}\frac{h\prime}{h}(\rho_m + \rho_r). \tag{7.9}$$

Let us assume that matter and radiation are decoupled then (7.9) gives with the initial conditions at present time $t_0 = 0$

$$a(t_0) = h(t_0) = 1 \tag{7.10}$$

the solutions

$$\rho_m = \rho_{m0}/\sqrt{h}, \quad \rho_r = \rho_{r0}/\left(a\sqrt{h}\right) \tag{7.11}$$

where ρ_{m0} and ρ_{r0} are the densities at present time $t_0 = 0$.

The initial conditions for the differential equations (7.4) are (see (7.10))

$$a(t_0) = h(t_0) = 1, \mathrm{a}'(t_0) = H_0, h'(t_0) = h'_0 \tag{7.12}$$

where H_0 is the Hubble constant and h'_0 is a further constant which does not arise by the use of the theory of general relativity.

Put

$$\varphi_0 = 3H_0\left(1 + \frac{1}{6}\frac{h\prime_0}{H_0}\right) \tag{7.13}$$

Then, it follows from (7.4) and (7.5) by the use of the initial conditions (7.12)

$$\frac{h\prime}{h} = -6\frac{a'}{a} + 2\frac{4\kappa c^4\lambda t + \varphi_0}{2\kappa c^4\lambda t^2 + \varphi_0 t + 1}. \tag{7.14a}$$

Further integration with the initial conditions (7.12) yields

$$a^3\sqrt{h} = 2\kappa c^4\lambda t^2 + \varphi_0 t + 1. \tag{7.14b}$$

Relation (7.8) gives with $t = t_0$

$$\frac{1}{3}(8\kappa c^4\lambda - \varphi_0^2) = 4\left(\frac{8}{3}\pi k\left(\rho_{m0} + \rho_{r0} + \frac{\Lambda c^2}{8\pi k}\right) - H_0^2\right). \tag{7.15}$$

We define as usually the density parameters

$$\Omega_m = \frac{8\pi k\rho_{m0}}{3H_0^2}, \Omega_r = \frac{8\pi k\rho_{r0}}{3H_0^2}, \Omega_\Lambda = \frac{\Lambda c^2}{3H_0^2} \tag{7.16}$$

and put

$$\Omega_m K_0 = \Omega_m + \Omega_r + \Omega_\Lambda - 1. \tag{7.17}$$

It follows from (7.8) by the use of (7.14), (7.17) and the elimination of h and h' in (7.8) the differential equation

$$\left(\frac{a'}{a}\right)^2 = \frac{H_0^2}{(2\kappa c^4 \lambda t^2 + \varphi_0 t + 1)^2}\{-\Omega_m K_0 + \Omega_r a^2 + \Omega_m a^3 + \Omega_\Lambda a^6\}. \tag{7.18}$$

Relation (7.15) is rewritten by the use of (7.16) and (7.17):

$$\frac{2\kappa c^4 \lambda}{H_0^2} - \left(\frac{1}{2}\frac{\varphi_0}{H_0}\right)^2 = 3\Omega_m K_0. \tag{7.19}$$

It follows from (7.19) that

$$K_0 > 0 \tag{7.20}$$

is equivalent to

$$2\kappa c^4 \lambda t^2 + \varphi_0 t + 1 > 0 \tag{7.21}$$

for all $t \in]-\infty, +\infty[$. Hence, we have from (7.18) and (7.14b) that (7.20) is necessary and sufficient for the existence of non-singular cosmological models.

Hence, the sum of the density parameters (7.17) is greater than one. Therefore, the solutions $a(t)$ of (7.18) and $h(t)$ given by (7.14b) describe a homogeneous, isotropic model of the universe.

We get from (7.18) that

$$a(t) \geq a_1 > 0 \tag{7.22a}$$

for all $t \in]-\infty, +\infty[$ where a_1 is defined by

$$\Omega_\Lambda a_1^6 + \Omega_m a_1^3 + \Omega_r a_1^2 - \Omega_m K_0 = 0. \tag{7.22b}$$

We require

$$K_0 \ll 1$$

to get small values for $a(t)$.In the following, let us assume

$$\rho_{r0} = 0. \tag{7.23}$$

Under the condition (7.23) an analytic solution of (7.18) can be given. Were write the equation (7.18) with initial condition (7.10) in the form

$$\frac{a\prime}{a} = \pm \frac{H_0}{2\kappa c^4 \lambda t^2 + \varphi_0 t + 1}(-\Omega_m K_0 + \Omega_m a^3 + \Omega_\Lambda a^6)^{1/2}, a(0) = 1. \quad (7.24)$$

The upper (lower) sign implies an increasing (decreasing) of the function $a(t)$. Standard integration methods and some trigonometric addition theorems give after longer calculations for the upper sign the solution

$$a^3(t) = 2K_0 / \left\{1 - (1 - 2K_0) cos\left(\sqrt{3}\alpha(t)\right) - 2\left(\frac{K_0}{\Omega_m}\right)^{1/2} sin\left(\sqrt{3}\alpha(t)\right)\right\} \quad (7.25a)$$

where

$$\alpha(t) = arctg\left\{(3\Omega_m K_0)^{1/2} H_0 t / \left(1 + \frac{1}{2}\varphi_0 t\right)\right\}. \quad (7.25b)$$

The detailed derivation of the result (7.25) is found in the article of [Pet 90].

We will now calculate the time t_1 where $a(t)$ reaches its minimal value a_1.This follows from

$$a'(t_1) = 0,$$

i.e., by the use of relation (7.18)

$$(1 - 2K_0) sin\left(\sqrt{3}\alpha(t_1)\right) - 2\left(\frac{K_0}{\Omega_m}\right)^{1/2} cos\left(\sqrt{3}\alpha(t_1)\right) = 0. \quad (7.26)$$

We get by elementary considerations

$$H_0 t_1 = -1 / \left\{\frac{1}{2}\frac{\varphi_0}{H_0} - \frac{(3\Omega_m K_0)^{1/2}}{A}\right\} \quad (7.27a)$$

with

$$A = tg\left\{\frac{1}{\sqrt{3}}\left(-\pi + arctg\left[2\frac{(K_0/\Omega_m)^{1/2}}{1 - K_0}\right]\right)\right\} \approx 4.03 + O(\sqrt{K_0}). \quad (7.27b)$$

A small value of the function $a(t)$ in the early universe yields

$$0 < K_0 \ll 1. \quad (7.28)$$

To get a solution $a(t)$ which goes to infinity as $t \to \infty$ the denominator of (7.25a) must go to zero. This implies

$$\frac{1}{2}\frac{\varphi_0}{H_0} \approx \frac{3}{2}\Omega_m/\left(1-\sqrt{\Omega_\Lambda}\right) \tag{7.29}$$

where expressions containing K_0 are omitted by virtue of (7.28).

We will now state a simple asymptotic formula for $t \to \infty$.

Define $\tilde{t}_1$ by

$$H_0\tilde{t}_1 = -\frac{1}{2}\frac{\varphi_0}{H_0}/(\left(\frac{1}{2}\frac{\varphi_0}{H_0}\right)^2 + 3\Omega_m \mathrm{K}_0).$$

It follows from (7.25) by longer elementary calculations the asymptotic representation (see e.g. $|Pet\ 98b|$)

$$a^3(t) \approx \frac{3}{4}\left(\frac{\Omega_m}{1-\sqrt{\Omega_\Lambda}}\right)^2 (H_0t - H_0\tilde{t}_1)^2/\left\{\sqrt{\Omega_\Lambda}(H_0t - H_0\tilde{t}_1) + \frac{1}{3}\frac{\left(1-\sqrt{\Omega_\Lambda}\right)^2}{\Omega_m}\right\}$$

$$\sqrt{h(t)} \approx 3\sqrt{\Omega_\Lambda}(H_0t - H_0\tilde{t}_1) + \frac{\left(1-\sqrt{\Omega_\Lambda}\right)^2}{\Omega_m}.$$

Let us assume that t_2 is as large such that the above asymptotic formulae hold and denote by τ_2 the corresponding proper time. Then, we get

$$\tau = \tau_2 + \int_{t_2}^{t}\frac{1}{\sqrt{h(t)}}dt \approx \tau_2 + \frac{1}{3\sqrt{\Omega_\Lambda}}\frac{1}{H_0}\log\left\{\frac{\sqrt{h(t)}}{\sqrt{h(t_2)}}\right\}.$$

This relation implies the asymptotic density of matter

$$\rho_m(\tau) = \frac{\rho_{m0}}{\sqrt{h(t)}} \approx \frac{\rho_{m0}}{\sqrt{h(t_2)}} exp\left\{-3\sqrt{\Omega_\Lambda}H_0(\tau - \tau_2)\right\}.$$

Hence, we get an exponential decay of matter in analogy to the radio-active decay.

The case $\Omega_\Lambda = 0$ must be considered separately by virtue of (7.17) with (7.28) and will be studied in sub-chapter 7.2.

The solution of (7.18) with (7.23) and the initial condition $a(t_1) = a_1$ can also be given. It holds

$$a^3(t) = 2a_1^3\left(1 + \frac{\Omega_\Lambda}{\Omega_m}a_1^3\right)\Big/\left(1 + \left(1 + 2\frac{\Omega_\Lambda}{\Omega_m}a_1^3\right)cos\left(\sqrt{3}\beta(t)\right)\right) \tag{7.30a}$$

for all $t \in\ ]-\infty, +\infty[$. Here, it holds

$$\beta(t) = arctg\left\{\frac{\sqrt{3\Omega_m K_0}H_0(t-t_1)}{\left(1+\frac{1}{2}\varphi_0 t_1+\left[\left(\left(\frac{1\varphi_0}{2H_0}\right)^2+3\Omega_m K_0\right)H_0 t_1+\frac{1\varphi_0}{2H_0}\right]H_0 t\right)}\right\} \tag{7.30b}$$

The relations (7.30) yield as $t \to -\infty$

$$a(-\infty) \approx \left(2/\left(1-cos\left(\sqrt{3}\pi\right)\right)\right)^{1/3} a_1 \approx 1.81 a_1. \tag{7.31a}$$

Hence, the function $a(t)$ starts at $t = -\infty$ from 1.81 a_1, decreases to a_1 and then increases to infinity as t goes to infinity. It follows from (7.14) by the use of (7.31a) as $t \to -\infty$:

$$\sqrt{h} \approx \left(\frac{1}{2}\frac{\varphi_0}{H_0}t+1\right)^2 / a^3(-\infty). \tag{7.31b}$$

The function $h(t)$ starts from infinity at $t = -\infty$, decreases to a positive value and then increases to infinity as t goes to infinity.

Therefore, we have for all $t \in]-\infty, +\infty[$

$$a(t) \geq a_1 > 0. \tag{7.32}$$

Hence, we have non-singular cosmological models by virtue of (7.11). In the beginning of the universe there are no matter and no vacuum energy, which is given by the cosmological constant, i.e. all the energy is in form of gravitational energy. In the course of time matter and radiation arises at coasts of gravitational energy. After a certain time the energy of matter and of radiation decreases and again go to zero as in the beginning of the universe. Therefore, in contrast to general relativity there is no singularity, i.e. no big bang.

The second law of thermodynamics is given by

$$dU = -PdV + TdS \tag{7.33}$$

where U, P, V, T and S denote energy, pressure, volume, absolute temperature and entropy.

The conservation of the whole energy (7.8) gives with

$$U = C\lambda c^2 \tag{7.34}$$

with a suitable constant C the relation

$$\frac{d}{dt}U = 0. \tag{7.35}$$

The comparison of (7.35) and (7.33) yields

$$PdV = TdS. \tag{7.36}$$

The right hand side of equation (7.36) is by the third law of thermodynamics non-negative. For an expanding space it must hold $P \geq 0$. Here, P is the pressure of the gravitational field and of the field implied by the cosmological constant. It follows by the use of the asymptotic formulae for a^3 and $\sqrt{h}$ that this condition is at least fulfilled for sufficiently large times. In the beginning of the universe space is contracting by virtue of (7.24). Hence, for $t \to -\infty$ it must hold $P \leq 0$. The pressure of the gravitational field dominates the other ones. We get by the use of (7.31):

$$P \approx \frac{1}{2\kappa c^2}\left(\frac{1}{2}\frac{\varphi_0}{H_0}\right)^2 H_0^2 > 0$$

which contradicts the condition for $t \to -\infty$. These considerations also hold for the case of $\Lambda = 0$ because the cosmological constant is not important in the beginning of the universe.

The application of equation (7.9) to the third law of thermodynamics requires a non-increasing function $h(t)$ to get entropy production in contradiction to the increasing of this function. Hence, we see that for the case $\Lambda > 0$ the universe is not expanding and not contracting and there is no entropy production.

Cosmological models without singularities by the use of flat space-time theory of gravitation are already studied in the article $[Pet\ 81b]$.

7.2 Homogeneous Isotropic Cosmological Model without Cosmological Constant

In this sub-chapter cosmological models with $\Lambda = 0$ and without loss of generality with put $\Omega_r = 0$. We start with the previous section under the assumption $\Omega_\Lambda = 0$. We get byequation (7.22b)

$$K_0 = a_1^3. \tag{7.37}$$

It follows from (7.25) as $t \to \infty$ by longer elementary calculations and the use of (7.28) and (7.13)

$$\frac{h_0{}'}{H_0} = -\frac{2}{3}K_0 < 0. \tag{7.38}$$

Again the considerations of (7.33) to (7.36) hold implying a non-expanding universe. But we may also start from the relation (7.9) by multiplication with a constant C. It follows

$$dU = -\frac{\rho_{r0}}{\left(ah^{1/8}\right)^4} dV - \frac{1}{2} C(\rho_m + \rho_r) \frac{h\prime}{h} dt. \tag{7.39}$$

By the use of the black body temperature

$$T = T_0/\left(ah^{1/8}\right) \tag{7.40}$$

relation (7.39) has again the form of the second law of thermodynamics (7.33) with

$$\begin{gathered} U = C(\rho_m + \rho_r),\ V = Ca^3,\ \ P = \rho_{r0}\left(\frac{T}{T_0}\right)^4, \\ dS = -\frac{1}{2} C(\rho_m + \rho_r) \frac{ah^{1/8}}{T_0} \frac{h\prime}{h} dt. \end{gathered} \tag{7.41}$$

Relation (7.41) implies by the use of the second law of thermodynamics that $h(t)$ must decrease for all $t \in]-\infty, +\infty[$ to give entropy production. This is at present time$t_0$stated by (7.38). Hence, in the case $\Lambda = 0$ the interpretation of a contracting and then expanding universe is possible. The time t_1 is again calculated as before by $a'(t_1) = 0$ of the solution (7.25). It follows by longer calculations and condition (7.28):

$$H_0 t_1 \approx -\frac{2}{3}\left\{1 + \frac{3}{2} \frac{\sqrt{3K_0}}{tg\left(-\pi/\sqrt{3}\right)}\right\}. \tag{7.42a}$$

We get by the use of (7.14b) with (7.19) and (7.37)

$$h(t_1)^{1/2} \approx \frac{4}{3}\left(1 + 1/tg^2\left(-\pi/\sqrt{3}\right)\right) \approx 1.41. \tag{7.42b}$$

Therefore, the creation of matter given by (7.11) from the minimal value of $a(t)$ at t_1 till the present time $t_0 = 0$ is given by a factor of about 1.4. It follows from (7.17)

$$\Omega_m = 1/(1 - K_0) \approx 1 + K_0. \tag{7.43}$$

The density parameter of matter Ω_m is a little bit greater than one.

The epoch before t_1, i.e. $t \in]-\infty, t_1[$ can be received by the results of chapter 7.1.

All these results are contained in the articles of Petry [Pet $90a$] and [Pet $90b$].

We will now give an asymptotic solution as $t \to \infty$. Put

$$H_0\tilde{t}_1 = -1/\left(\frac{1}{2}\frac{\varphi_0}{H_0}\right) \tag{7.44a}$$

and assume that

$$H_0 t \gg H_0\tilde{t}_1. \tag{7.44b}$$

Then, the differential equation (7.18) with $\Omega_\Lambda = 0$ has the form

$$\frac{a\prime}{a} \approx H_0\sqrt{\Omega_m}a(t)^{3/2}\Big/\left(\frac{1}{2}\frac{\varphi_0}{H_0}(H_0 t - H_0\tilde{t}_1)\right)^2. \tag{7.45a}$$

The solution is given by

$$a^3(t) \approx \frac{4}{9}\frac{1}{\Omega_m}\left(\frac{1}{2}\frac{\varphi_0}{H_0}\right)^4 (H_0 t - H_0\tilde{t}_1)^2. \tag{7.45b}$$

The study of $a(t)$ for $t \to \infty$ given by (7.25) and the use of (7.13) and (7.28) imply

$$h(t)^{1/2} \approx \frac{9}{4}\Omega_m/\left(\frac{2\kappa c^4\lambda}{H_0^2}\right) \approx 1 - \frac{1}{9}K_0. \tag{7.46}$$

Hence, the function $h(t)$ is in the above stated region nearly constant and converges to a positive value which is a little bit smaller than one. Therefore, the function $a(t)$ starts from a finite positive value $a(-\infty)$ and $h(t)$ from infinity. In the course of time $a(t)$ decreases till to the time t_1 with a value $a(t_1) = a_1 > 0$. After that time $a(t)$ always increases to infinity. The function $h(t)$ decreases for all times to a positive value which is a little bit smaller than the present value in agreement with the third law of thermodynamics stated by relation (7.9). Hence, in the case that the cosmological constant is zero the interpretation of an expanding space is permitted but also the interpretation of a non-expanding space is possible by the considerations of sub-section 7.1.

Furthermore, it appears that at present time the universe is nearly stationary. In the final state only matter and gravitational energy exist.

It is worth mentioning that in addition to matter, radiation and cosmological constant a further kind of energy may be introduced in the study of cosmological models. Such considerations can be found in the articles [Pet $94c$] and [Pet $98a$].

A cosmological model with a scaling dependent cosmological constant is studied in the article [Pet 08].

Summarizing, the cosmological models of flat space-time theory of gravitation are in the beginning of the universe quite different from those of general relativity. In the beginning strong gravitational fields exist. The received models are non-singular, i.e. a big bang does not exist. Formula (7.45b) is identical with the result of general relativity, i.e. for sufficiently large times after the minimum of the universe the two theories give the same result.