Chapter 5

Post-Newtonian Approximation

In this chapter post-Newtonian approximation of the gravitational field in flat space-time of a perfect fluid is studied. The conservation laws of energy-momentum and of angular-momentum are derived. The equivalence of the conservation law of energy-momentum and of the equations of motion is shown to the studied accuracy. All the results of post-Newtonian approximation in flat space-time theory of gravitation agree up to the studied accuracy with those of general relativity as studied by Will in his famous book of Will [Wil 81].

5.1 Post-Newtonian Approximation

The study of post-Newtonian approximation of gravitation in flat space-time follows along the considerations of Will. In this sub-chapter we assume a matter tensor of the form

$$T(M)_i^j = \left(\frac{-G}{-\eta}\right)^{1/2} \left\{\left(\rho\left(1 + \frac{\Pi}{c^2}\right) + p\right) g_{ik} \frac{dx^k}{d\tau}\frac{dx^j}{d\tau} + pc^2 \delta_i^j\right\} \tag{5.1a}$$

where ρ denotes the density of matter, Π is the specific internal energy, p is the isotropic pressure and $\left(\frac{dx^i}{d\tau}\right)$ is the four-velocity. Equation (5.1a) yields by the use of relation (1.8)

$$T(M)_k^k = -\left(\frac{-G}{-\eta}\right)^{1/2} \left(\rho\left(1 + \frac{\Pi}{c^2}\right) - 3p\right) c^2. \tag{5.1b}$$

The post-Newtonian approximation is an expansion of the gravitational field in powers of $\frac{1}{c}$. Subsequently, we use the pseudo-Euclidean geometry given by (1.4) and (1.5). Let us start with the Newtonian gravitational potential defined by

$$\Delta U = -4\pi k\rho \tag{5.2a}$$

with the solution

$$U(x,t) = k \int \frac{\rho(x',t)}{|x-x\prime|} dx'^3. \tag{5.2b}$$

Sub-chapter 2.2 implies the approximate tensor

$$
\begin{aligned}
g_{ij} &= 1 + \frac{2}{c^2}U(i = j = 1,2,3) \\
&= -\left(1 - \frac{2}{c^2}U\right)(i = j = 4) \\
&= 0(i \neq j)
\end{aligned}
\tag{5.3a}
$$

with the inverse tensor

$$
\begin{aligned}
g^{ij} &= 1 - \frac{2}{c^2}U(i = j = 1,2,3) \\
&= -\left(1 + \frac{2}{c^2}U\right)(i = j = 4) \\
&= 0(i \neq j)
\end{aligned}
\tag{5.3b}
$$

Let v denote the velocity of the body, i.e.

$$
v = (v^1, v^2, v^3) = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt}, \frac{dx^3}{dt}\right)
\tag{5.4}
$$

And assume in analogy to Will

$$
\left|\frac{v}{c}\right|^2 \sim \frac{1}{c^2}U \sim \frac{p}{\rho} \sim \frac{\Pi}{c^2} \sim O\left(\frac{1}{c^2}\right)
\tag{5.5a}
$$

and

$$
\left|\frac{\partial/\partial t}{\partial/\partial x^i}\right| \sim O(1).
\tag{5.5b}
$$

The post-Newtonian approximation of gravitation now requires the knowledge of g_{44} to $O\left(\frac{1}{c^4}\right)$, of g_{i4} to $O\left(\frac{1}{c^3}\right)$ and of g_{ij} to $O\left(\frac{1}{c^2}\right)$ (i,j=1,2,3). Hence, we make the ansatz

$$
\begin{aligned}
g_{ij} &= \left(1 + \frac{2}{c^2}U\right)\delta_{ij}(i, j = 1,2,3) \\
&= -\frac{4}{c^3}V_i(i = 1,2,3; j = 4) \\
&= -\frac{4}{c^3}V_j(i = 4; j = 1,2,3) \\
&= -\left(1 - \frac{2}{c^2}U + \frac{1}{c^4}S\right)(i = j = 4)
\end{aligned}
\tag{5.6a}
$$

with

$$V_i \sim S \sim O(1). \tag{5.6b}$$

The inverse tensor of (5.6a) is

$$\begin{aligned} g^{ij} &= \left(1-\frac{2}{c^2}U\right)\delta^{ij}\,(i;\, j=1,2,3) \\ &= -\frac{4}{c^3}V_i\,(i=1,2,3;\, j=4) \\ &= -\frac{4}{c^3}V_j\,(i=4;\, j=1,2,3) \\ &= -\left(1+\frac{2}{c^2}U+\frac{1}{c^4}(-S+4U^2)\right)(i=j=4) \end{aligned} \tag{5.6c}$$

In addition, we have

$$\left(\frac{-G}{-\eta}\right)^{1/2} \approx 1+\frac{2}{c^2}U \tag{5.6d}$$

It follows from (1.13) and (1.12) by the use of (5.4) and (5.6)

$$\frac{dt}{d\tau} \approx 1+\frac{1}{c^2}U+\frac{1}{2}\left|\frac{v}{c}\right|^2. \tag{5.7}$$

We get from (5.1) with the aid of (5.6) and (5.7)

$$\begin{aligned} T(M)^i_j &= \rho v^i v^j + pc^2\delta^i_j \quad (\mathrm{i};\ \mathrm{j}=1,2,3) \\ &= \rho c v^i\left(1+\frac{\Pi}{c^2}+\frac{6}{c^2}U+\left|\frac{v}{c}\right|^2+\frac{p}{\rho}\right)-\frac{4}{c}\rho V_j \\ &\qquad (\mathrm{i}=4;\ \mathrm{j}=1,2,3) \\ &= -\rho c v^i\left(1+\frac{\Pi}{c^2}+\frac{2}{c^2}U+\left|\frac{v}{c}\right|^2+\frac{p}{\rho}\right) \quad (\mathrm{i}=1,2,3;\ \mathrm{j}=4) \\ &= -\rho c^2\left(1+\frac{\Pi}{c^2}+\frac{2}{c^2}U+\left|\frac{v}{c}\right|^2\right) \quad (\mathrm{i}=\mathrm{j}=4) \end{aligned} \tag{5.8a}$$

to $O(1)$ and $O\left(\frac{1}{c}\right)$ respectively. Furthermore, we get to $O(1)$

$$T(M)^k_k = -\rho c^2\left(1+\frac{\Pi}{c^2}+\frac{2}{c^2}U-3\frac{p}{\rho}\right). \tag{5.8b}$$

We have from (1.21a) and (1.9) by the use of (5.6) the mixed energy-momentum tensor of the gravitational field to the same accuracy as that of matter

$$\begin{aligned} T(G)_j^i &= \frac{1}{8\kappa}\frac{8}{c^4}\left(\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^j} - \frac{1}{2}\delta_j^i \sum_{k=1}^{3}\frac{\partial U}{\partial x^k}\frac{\partial U}{\partial x^k}\right) \quad \text{(i; j=1,2,3)} \\ &= -\frac{1}{8\kappa}\frac{8}{c^4}\frac{\partial U}{\partial ct}\frac{\partial U}{\partial x^j} \quad \text{(i=4; j=1, 2, 3)} \\ &= +\frac{1}{8\kappa}\frac{8}{c^4}\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial ct} \quad \text{(i=1, 2, 3; j=4)} \\ &= -\frac{1}{8\kappa}\frac{4}{c^4}\sum_{k=1}^{3}\frac{\partial U}{\partial x^k}\frac{\partial U}{\partial x^k} \quad \text{(i=j=4)} \end{aligned} \tag{5.9a}$$

and

$$T(G)_l^l = -\frac{1}{8\kappa}\frac{8}{c^4}\sum_{k=1}^{3}\frac{\partial U}{\partial x^k}\frac{\partial U}{\partial x^k}. \tag{5.9b}$$

We now obtain from (1.24) with the aid of (5.6), (5.8), (5.9) and (5.2) by longer elementary calculations

$$\Delta V_i = -4\pi k\rho v^i \quad \text{(i=1,2,3)} \tag{5.10a}$$

And

$$\Delta S - 4\sum_{k=1}^{3}\frac{\partial}{\partial x^k}\left(U\frac{\partial U}{\partial x^k}\right) + 2\frac{\partial^2 U}{\partial t^2} = 8\pi k\rho\left(\Pi + 2U + 2|v|^2 + 3\frac{pc^2}{\rho}\right). \tag{5.10b}$$

Here, (5.10a) follows with i=1,2,3; j=4 (or i and j exchanged) and equation (5.10b) with i=j=4. The equations (1.23) with i, j=1,2,3 are identically satisfied by virtue of (5.2).The solution of (5.10a) is given by

$$V_i = k\int\frac{\rho' v^{i\prime}}{|x-x'|}d^3x' \quad \text{(i=1,2,3)} \tag{5.11}$$

where $\rho' = \rho(x', t)$ and correspondingly $v^{i'}$.To solve equation (5.10b) we use the identity

$$\Delta U^2 = 2\sum_{k=1}^{3}\frac{\partial}{\partial x^k}\left(U\frac{\partial U}{\partial x^k}\right)$$

and introduce in analogy to Chandrasekhar the super-potential

$$\chi = -k\int\rho'|x-x'|\,d^3x' \tag{5.12a}$$

which satisfies

$$\triangle \chi = -2U. \tag{5.12b}$$

Hence, the equation (5.10b) can be rewritten

$$\triangle \left(S - 2U^2 - \frac{\partial^2 \chi}{\partial t^2}\right) = 8\pi k\rho \left(\Pi + 2U + 2|v|^2 + 3\frac{pc^2}{\rho}\right). \tag{5.13}$$

Furthermore, let us put $(see\ |Wil\ 81|)$

$$\begin{aligned} \phi_1 &= k\int \frac{\rho'|v'|^2}{|x-x'|} d^3x', \phi_2 = k\int \frac{\rho' U'}{|x-x'|} d^3x', \\ \phi_3 &= k\int \frac{\rho'\Pi'}{|x-x'|} d^3x', \phi_4 = k\int \frac{p'}{|x-x'|} d^3x'. \end{aligned} \tag{5.14a}$$

and

$$\phi = 2\phi_1 + 2\phi_2 + \phi_3 + 3\phi_4 \tag{5.14b}$$

then, the equation (5.13) has the solution

$$S = 2U^2 + \frac{\partial^2 \chi}{\partial t^2} - 2\phi. \tag{5.15}$$

Hence, the tensors (g_{ij}) and (g^{ij}) of (5.6a) and (5.6c) are known to the needed accuracy. Will $[Wil\ \ 81]$ has shown that any metric theory of gravitation may be given by a suitable transformation in the so-called "standard form". For the metric (5.6a) this transformation is given by

$$c\tilde{t} = ct - \frac{1}{2c^3}\frac{\partial \chi}{\partial t},$$

i.e. only by a time-transformation. But it will be shown that there is no necessity for such a transformation as already remarked by Chugreev $[Chu\ 90]$.

5.2 Conservation Laws

When we start instead of (5.6a) from the better approximation for i, j =1,2,3

$$g_{ij} = \left(1 + \frac{2}{c^2}U\right)\delta_{ij} + \frac{1}{c^4}S_{ij}$$

where $S_{ij} = O(1)$ then the energy-momentum tensor (1.21a) can be calculated to the accuracy

$$T(G)^i_j \sim O\left(\frac{1}{c^2}\right) \quad (i, j = 1,2,3)$$
$$\sim O\left(\frac{1}{c}\right) \quad (i=1, 2, 3; j=4), \ (i=4; j=1, 2, 3) \qquad (5.16)$$
$$\sim O(1) \quad (i=j=4).$$

Elementary calculations give

$$T(G)^i_j = \frac{1}{\kappa c^4}\left(\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^j} + \frac{4}{c^2}U\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial x^j} - \frac{4}{c^2}\sum_{k=1}^{3}\frac{\partial V_k}{\partial x^i}\frac{\partial V_k}{\partial x^j}\right.$$
$$\left. - \frac{1}{2c^2}\left(\frac{\partial U}{\partial x^i}\frac{\partial S}{\partial x^j} + \frac{\partial U}{\partial x^j}\frac{\partial S}{\partial x^i}\right) + \delta^i_j\frac{c^4}{16}L_G\right) \quad (i, j=1,2,3)$$
$$= \frac{1}{\kappa c^4}\frac{\partial U}{\partial x^i}\frac{\partial U}{\partial ct} \quad (i=1, 2, 3; j=4) \qquad (5.17a)$$
$$= -\frac{1}{\kappa c^4}\frac{\partial U}{\partial ct}\frac{\partial U}{\partial x^j} \quad (i=4; j=1, 2, 3)$$
$$= -\frac{1}{2\kappa c^4}\sum_{k=1}^{3}\frac{\partial U}{\partial x^k}\frac{\partial U}{\partial x^k} \quad (i=j=4)$$

where

$$\frac{c^4}{8}L_G = -\sum_{k=1}^{3}\left(\frac{\partial U}{\partial x^k}\right)^2 + \left(\frac{\partial U}{\partial ct}\right)^2 - \frac{4}{c^2}U\sum_{k=1}^{3}\left(\frac{\partial U}{\partial x^k}\right)^2$$
$$+\frac{4}{c^2}\sum_{k=1}^{3}\left(\frac{\partial V_l}{\partial x^k}\right)^2 + \frac{1}{c^2}\sum_{k=1}^{3}\frac{\partial U}{\partial x^k}\frac{\partial S}{\partial x^k}. \qquad (5.17b)$$

Hence, the energy-momentum tensor $T(G)^i_j$ of (5.17) is given to the stated accuracy (5.16). It follows that the knowledge of S_{ij} is not necessary.

We will now calculate $T(M)^{ij}$ to the same accuracy as stated by (5.16). It follows from (1.28), (5.4), (5.6) and (5.7) for the symmetric matter tensor

$$T(M)^{ij} = \rho\left(1 + \frac{\Pi}{c^2} + \frac{4}{c^2}U + \left|\frac{v}{c}\right|^2 + \frac{p}{\rho}\right)v^i v^j + pc^2\delta^{ij} \ (i, j = 1,2,3)$$
$$= \rho\left(1 + \frac{\Pi}{c^2} + \frac{4}{c^2}U + \left|\frac{v}{c}\right|^2 + \frac{p}{\rho}\right)cv^i \ (i = 1,2,3; j = 4) \qquad (5.18)$$
$$= \rho\left(1 + \frac{\Pi}{c^2} + \frac{4}{c^2}U + \left|\frac{v}{c}\right|^2\right)c^2. (i = j = 4)$$

We obtain from (5.17) by the use of (5.13), (5.2a) and (5.10a)

$$\begin{aligned}\frac{\partial}{\partial x^k}T(G)_j^k = -\rho\left(1+\frac{\Pi}{c^2}+\frac{4}{c^2}U+2\left|\frac{v}{c}\right|^2+3\frac{p}{\rho}\right)\frac{\partial U}{\partial x^j} \\ +\frac{4}{c^2}\rho\sum_{k=1}^{3}v^k\frac{\partial V_k}{\partial x^j}+\frac{1}{2c^2}\rho\frac{\partial S}{\partial x^j}\end{aligned} \tag{5.19a}$$

to accuracy of $O\left(\frac{1}{c^2}\right)$ and

$$\frac{\partial}{\partial x^k}T(G)_4^k = -\frac{1}{c}\rho\frac{\partial U}{\partial t} \tag{5.19b}$$

to accuracy of $O\left(\frac{1}{c}\right)$. It follows from (5.18) and (5.6a)

$$\begin{aligned}\frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}T(M)^{kl} = \rho\left(1+\frac{\Pi}{c^2}+\frac{4}{c^2}U+2\left|\frac{v}{c}\right|^2+3\frac{p}{\rho}\right)\frac{\partial U}{\partial x^j} \\ -\frac{4}{c^2}\rho\sum_{k=1}^{3}v^k\frac{\partial V_k}{\partial x^j}-\frac{1}{2c^2}p\frac{\partial S}{\partial x^j}\ (j=1,2,3)\end{aligned} \tag{5.20a}$$

to accuracy $O\left(\frac{1}{c^2}\right)$ and

$$\frac{1}{2}\frac{\partial g_{kl}}{\partial ct}T(M)^{kl} = \frac{1}{c}\rho\frac{\partial U}{\partial t} \tag{5.20b}$$

to accuracy $O\left(\frac{1}{c}\right)$. Hence, we get by comparing (5.19) and (5.20)

$$\frac{\partial}{\partial x^k}T(G)_j^k = -\frac{1}{2}\frac{\partial g_{kl}}{\partial x^j}T(M)^{kl} \quad \text{(j=1-4)} \tag{5.21}$$

to accuracy $O\left(\frac{1}{c^2}\right)$ for j=1,2,3 and to $O\left(\frac{1}{c}\right)$ for j=4. The equations of motion (1.29a) to the above noted accuracy are equivalent to the conservation law of energy-momentum (see (1.25a))

$$\frac{\partial}{\partial x^k}\left(T(G)_j^k + T(M)_j^k\right) = 0 \quad \text{(j=1-4)}. \tag{5.22}$$

Put

$$P_j = \int\left(T(G)_j^4 + T(M)_j^4\right)d^3x \quad \text{(j=1-4)}. \tag{5.23}$$

Hence, P_j is constant to accuracy $O\left(\frac{1}{c}\right)$ for j=1,2,3 and to accuracy $O(1)$ for j=4. It follows from (5.23) with the aid of (5.8a), (5.9a), (5.2a) and (5.12b) by the theorem of Gauß

$$P_j = c\int \rho\left\{v^j\left(1+\frac{\Pi}{c^2}+\frac{6}{c^2}U+\left|\frac{v}{c}\right|^2+\frac{p}{\rho}\right)-\frac{4}{c^2}V_j+\frac{1}{2c^2}\frac{\partial^2\chi}{\partial t\partial x^j}\right\}d^3x \quad (5.24a)$$

for j=1,2.3 and

$$P_4 = -c^2\int \rho\left(1+\frac{\Pi}{c^2}+\frac{5}{2c^2}U+\left|\frac{v}{c}\right|^2\right)d^3\,x \quad (5.24b)$$

where the identity

$$\sum_{k=1}^{3}\frac{\partial U}{\partial x^k}\frac{\partial U}{\partial x^k} = -U\Delta U+\sum_{k=1}^{3}\frac{\partial}{\partial x^k}\left(U\frac{\partial U}{\partial x^k}\right)$$

is used. Will $[Wil\ 81]$ introduces for j=1,2,3

$$W_j = k\int\frac{\rho\prime\left(v',(x-x\prime)\right)\left(x^j-x^j{}_\prime\right)}{|x-x\prime|^3}d^3x' \quad (5.25a)$$

then (compare also Chandrasekhar $[Cha\ 65]$)

$$\frac{\partial^2\chi}{\partial t\partial x^j} = V_j - W_j. \quad (5.25b)$$

We get from the conservation law for mass

$$\left(\left(\frac{-G}{-\eta}\right)^{1/2}\rho\frac{dx^k}{d\tau}\right)_{/k} = 0 \quad (5.26)$$

by the use of (5.4), (5.6d) and (5.7) the conservation law

$$\frac{\partial\rho^*}{\partial t}+\sum_{k=1}^{3}\frac{\partial}{\partial x^k}\left(\rho^* v^k\right) = 0 \quad (5.27a)$$

to $O\left(\frac{1}{c^2}\right)$ where

$$\rho^* = \rho\left(1+\frac{3}{c^2}U+\frac{1}{2}\left|\frac{v}{c}\right|^2\right). \quad (5.27b)$$

Hence, the conserved mass is given by

$$m = \int\rho^* d^3\,x. \quad (5.28)$$

The conserved energy-momentum follows from (5.24) with (5.27b) and by (5.25)

$$P_j = c\int \rho^* \left[v^j\left(1+\frac{\Pi}{c^2}+\frac{3}{c^2}U+\frac{1}{2}\left|\frac{v}{c}\right|^2+\frac{p}{\rho}\right) \ -\frac{1}{2c^2}\left(7V_j+W_j\right)\right]d^3x \quad (5.29a)$$

(j=1,2,3)

$$P_4 = -c^2\int \rho^*\left(1+\frac{\Pi}{c^2}-\frac{1}{2c^2}U+\frac{1}{2}\left|\frac{v}{c}\right|^2\right)d^3\,x. \quad (5.29b)$$

By the use of the identity (see e.g. [Cha 65])

$$\int \rho U v^j\, d^3x = \int \rho V_j\, d^3x$$

the momentum (5.29a) is rewritten to $O\left(\frac{1}{c}\right)$ in the form

$$P_j = c\int \rho^*\left[v^j\left(1+\frac{\Pi}{c^2}-\frac{1}{2c^2}U+\frac{1}{2}\left|\frac{v}{c}\right|^2+\frac{p}{\rho}\right)-\frac{1}{2c^2}W_j\right]d^3x. \quad (5.29c)$$

The conserved quantities of mass (5.28) and of the energy-momentum (5.29b) and (5.29c) are identical with the corresponding results of Einstein's theory (see [Cha 65] or [Wil 81]).

It is worth mentioning that we have used the energy-momentum tensor in the form (5.1a) with the factor $\left(\frac{-G}{-\eta}\right)^{1/2}$ to get formally the same results as those of general relativity. In general the above factor is omitted which would give the same results in another form of representation.

We will now study the conservation law of angular-momentum (1.53) in uniformly moving reference frames. We get

$$M^{ij} = \int\left(x^i\tilde{T}^{j4}-x^j\tilde{T}^{i4}+A^{ij4}\right)d^3x \quad (5.30)$$

is conserved for i, j=1,2,3,4. It follows by the use of (5.6) that $A^{ij4}=0$ to an accuracy of $O\left(\frac{1}{c}\right)$ for i, j=1,2,3 and to an accuracy of $O(1)$ for i=4; j=1,2,3 and i=1,2,3; j=4.

Hence, we obtain to the given accuracy the usual conservation law of angular-momentum, i.e. without spin expression:

$$M^{ij} = \int\left(x^i\tilde{T}^{j4}-x^j\tilde{T}^{i4}\right)d^3x. \quad (5.31)$$

In particular, for j=4 we get with (5.23)

$$M^{i4} = \int\left(-x^i T_4^4 - ctT_i^4\right)d^3x = -\int x^i\, T_4^4 d^3x - ctP_i. \tag{5.32}$$

If we substitute (5.8a) and (5.17a) into relation (5.32) we get for i=1,2,3 by elementary calculations

$$M^{i4} = c^2\int x^i \rho^*\left(1 + \frac{\Pi}{c^2} - \frac{1}{2c^2}U + \frac{1}{2}\left|\frac{v}{c}\right|^2\right)d^3x - ctP_i. \tag{5.33}$$

Defining the centre of the mass (X^1, X^2, X^3) (see Will [Wil 81]) by

$$X^i = \int x^i\rho^*\left(1 + \frac{\Pi}{c^2} - \frac{1}{2c^2}U + \frac{1}{2}\left|\frac{v}{c}\right|^2\right)d^3x \Big/ \int \rho^*\left(1 + \frac{\Pi}{c^2} - \frac{1}{2c^2}U + \frac{1}{2}\left|\frac{v}{c}\right|^2\right)d^3x.$$

We get from equation (5.33) by differentiation and the use of (5.29b)

$$\frac{d}{dt}X^i = -c\frac{P_i}{P_4} \quad \text{(i=1,2,3)} \tag{5.34}$$

i.e. the centre of the mass moves uniformly with the velocity $-\frac{c}{P_4}(P_1, P_2, P_3)$.

5.3 Equations of Motion

The equations of motion (1.29a) can be rewritten (see Petry [Pet 91])

$$\frac{\partial}{\partial x^k}T(M)^{jk} = -\Gamma(G)^j_{kl}T(M)^{kl}. \tag{5.35}$$

Elementary calculations give by the use of (5.6), (5.15) and (5.25b) for i, j, k=1,2,3 the Christoffel symbols

$$\begin{aligned}
&\Gamma(G)^4_{44} = -\frac{1}{c^3}\frac{\partial U}{\partial t}, \Gamma(G)^4_{4i} = -\frac{1}{c^2}\frac{\partial U}{\partial x^i},\\
&\Gamma(G)^4_{ij} = \frac{1}{c^3}\left\{\frac{\partial U}{\partial t}\delta_{ij} + 2\left(\frac{\partial V_i}{\partial x^j} + \frac{\partial V_j}{\partial x^i}\right)\right\},\\
&\Gamma(G)^i_{44} = -\frac{1}{c^2}\frac{\partial U}{\partial x^i} + \frac{1}{c^4}\left(2\frac{\partial U^2}{\partial x^i} - \frac{\partial \phi}{\partial x^i} - \frac{7}{2}\frac{\partial V_i}{\partial t} - \frac{1}{2}\frac{\partial W_i}{\partial t}\right),\\
&\Gamma(G)^i_{4j} = \frac{1}{c^3}\left\{\frac{\partial U}{\partial t}\delta_{ij} - 2\left(\frac{\partial V_i}{\partial x^j} - \frac{\partial V_j}{\partial x^i}\right)\right\},\\
&\Gamma(G)^i_{jk} = \frac{1}{c^2}\left(\frac{\partial U}{\partial x^k}\delta_{ij} + \frac{\partial U}{\partial x^j}\delta_{jk} - \frac{\partial U}{\partial x^i}\delta_{jk}\right).
\end{aligned} \tag{5.36}$$

The equations of motion are satisfied to accuracy $O\left(\frac{1}{c^2}\right)$ for j=1,2,3 and to accuracy $O\left(\frac{1}{c}\right)$ for j=4 (see (5.21)). Hence, it follows from formula (5.35) with j=4 that $\Gamma(G)^4_{ij} \approx 0$ (i, j=1,2,3) to the needed accuracy. Therefore, the Christoffel symbols (5.36) with $\Gamma(G)^4_{ij} = 0$ are identical with those of general relativity (see $[Wil\ 81]$ and $[Cha\ 65]$). The equations of motion (5.35) are by the use of (5.18) and (5.36) given to accuracy $O\left(\frac{1}{c^2}\right)$ for j=1,2,3 and to accuracy $O\left(\frac{1}{c}\right)$ for j=4. Here, the density ρ^* given by (5.27b) may be introduced instead of the densityρ.

Let $T(M_E)^{ij}$ denote the symmetric matter tensor of the theory of Einstein then we have the relation

$$T(M)^{ij} = \left(\frac{-G}{-\eta}\right)^{1/2} T(M_E)^{ij}. \tag{5.37}$$

The equations of motion of general relativity of Einstein can be written (see e.g. Fock $[Foc\ 60]$)

$$\frac{\partial}{\partial x^k}\left\{\left(-G(E)\right)^{1/2} T(M_E)^{ik}\right\} = -\Gamma(G_E)^i_{kl}\left(-G(E)\right)^{1/2} T(M_E)^{kl} \tag{5.38}$$

where $\Gamma(G_E)^i_{kl}$ are the Christoffel symbols of the theory of Einstein and $G(E)$ is the determinant of the corresponding metric. By virtue of (5.37), $\eta = -1$, $G = G(E)$ to $O\left(\frac{1}{c^2}\right)$ and the agreement of $\Gamma(G)^i_{jk}$ with $\Gamma(G_E)^i_{jk}$ to the needed accuracy the equations of motion (5.35) of gravitation in flat space-time agree with the equations of motion (5.38) of general relativity. Hence, the equations of motion are to post-Newtonian approximation identical with the results of the theory of Einstein.

Summarizing, all the results of flat space-time theory of gravitation and the general theory of relativity of Einstein agree to post-Newtonian approximation.

The results of this chapter on post-Newtonian approximations by the use of the theory of gravitation in flat space-time can be found in the article of Petry $[Pet\ 92]$. Post-Newtonian approximations to higher order (to $2\frac{1}{2}$) are given in the paper of Thümmel $[Th$ü $96]$ by the use of the theory of gravitation in flat space-time.