Chapter 1

Stochastic Volatility Models

Section 1.1

Heston Model: Option Pricing Formulae, Filtering and Calibration Problems

[Description] *We consider stock prices whose dynamics is described by the Heston model, that is by a system of two stochastic differential equations with a suitable initial condition. We estimate the parameters of the Heston model and one component of the initial condition, that is the initial stochastic variance, from the knowledge of the stock and option prices observed at discrete times. The method proposed to solve this problem is based on a filtering technique to construct a likelihood function and on the maximization of the likelihood function obtained. The estimated parameters and initial value component are characterized as being a maximizer of the likelihood function subject to some constraints. The solution of the filtering problem, used to construct the likelihood function, is based on an integral representation of the fundamental solution of the Fokker-Planck equation associated to the Heston model, on the use of a wavelet expansion to approximate the integral kernel appearing in the representation formula of the fundamental solution, on a simple truncation procedure to exploit the sparsifying properties of the wavelet expansions and on the use of the Fast Fourier Transform (FFT).*

[Paper] *Mariani F., Pacelli G., Zirilli F. (2008). Maximum likelihood estimation of the Heston stochastic volatility model using asset and option prices: an application of nonlinear filtering theory, Optimization Letters 2, 177-222.*

[Website] *http://www.econ.univpm.it/pacelli/mariani/finance/w1*

1.1.1 Outline of the Presentation

1. The Heston stochastic volatility model
   - The calibration problem
   - The estimation and filtering problems
   - Some numerical results
2. A stochastic volatility model for the index of the ”long short equity” hedge funds based on the Heston stochastic volatility model
   - The calibration problem
   - The estimation and filtering problems
   - Some numerical results
3. References

1.1.2 The Calibration Problem for the Heston Stochastic Volatility Model

We want to estimate the parameters of the Heston stochastic volatility model (Heston 1993) describing the dynamics of the stock log-returns starting from price data.

We use as data the observation at discrete times of the stock log-returns and of the prices of an European call option on the stock.

The solution approach that we propose makes use of:

- nonlinear filtering techniques,
- maximum likelihood method.

We focus on:

- the formulation of the problem;
- the accuracy of the solution;
- the computational efficiency of the solution method.

The problem considered is realistic and the analysis of time series of real financial data can be considered. Preliminary results are very promising.

1.1.3 The Heston Stochastic Volatility Model

Let x_t, $t > 0$, be the stock log-return at time t, v_t, $t > 0$, be the stochastic variance associated to the stock log-return x_t at time t.

We assume that the dynamics of x_t, $t > 0$, and of v_t, $t > 0$, is described by the Heston model :

$$\begin{aligned} dx_t &= \Big(\mu - \frac{1}{2}v_t\Big)dt + \sqrt{v_t}\, dW_t^1, \quad t > 0, \\ dv_t &= \gamma(\theta - v_t)dt + \varepsilon\sqrt{v_t}\, dW_t^2, \quad t > 0, \end{aligned}$$

where μ, γ, ε, θ are real constants, W_t^1, W_t^2, $t > 0$, are standard Wiener processes such that $W_0^1 = W_0^2 = 0$, dW_t^1, dW_t^2 are their stochastic differentials, $< dW_t^1 dW_t^2 >= \rho dt$, where $< \cdot >$ denotes the expected value of $\cdot$, and $\rho \in [-1, 1]$ is a constant known as correlation parameter.

We complete the Heston stochastic differential equations with the initial conditions:

$$x_0 = \tilde{x}_0,$$

$$v_0 = \tilde{v}_0,$$

where $\tilde{x}_0$ and $\tilde{v}_0$ denote random variables concentrated with probability one in a point that for simplicity we continue to denote with $\tilde{x}_0$ and $\tilde{v}_0$.

Remind that $\tilde{v}_0$ cannot be observed.

The parameters that must be estimated from the data are:

- the Heston model parameters: μ, γ, ε, θ, ρ,
- the initial stochastic variance: $\tilde{v}_0$,
- the risk premium parameter: λ (remind that we consider option prices as data and that we evaluate the option prices under the risk neutral measure, that is the probability measure associated to the Heston model where γ and θ are substituted respectively with $\gamma^* = \gamma + \lambda$ and $\theta^* = \gamma\theta/(\gamma + \lambda)$),

that is, the following vector: $\underline{\Theta} = (\mu, \gamma, \varepsilon, \theta, \rho, \tilde{v}_0, \lambda)^T$, where ^{T} denotes the transpose operator. Elementary considerations suggest that the following set of constraints must be satisfied by the vector $\underline{\Theta}$:

$$\mathcal{M} = \{\underline{\Theta} = (\mu, \gamma, \varepsilon, \theta, \rho, \tilde{v}_0, \lambda)^T \in \mathbb{R}^7 | \gamma \geq 0, \varepsilon \geq 0, \theta \geq 0,$$

$$\frac{2\gamma\theta}{\varepsilon^2} \geq 1, 1 \geq \rho \geq -1, \tilde{v}_0 \geq 0\}.$$

1.1.4 The Calibration Problem

DATA

- the observation times $0 = t_0 < t_1 < t_2 < \ldots < t_n < +\infty$;
- the stock log-return $\tilde{x}_i$, $i = 0, 1, 2, \ldots, n$;
- the option price $\tilde{C}_i$, $i = 0, 1, 2, \ldots, n$.

We assume:

1. $\tilde{x}_i = x_{t_i}$, $i = 0, 1, ..., n,$;
2. $\tilde{C}_i = C(\tilde{x}_i, \tilde{v}_i, t_i; E, T) + u_i$, $i = 0, 1, ..., n$,

where

1. $C(x, v, t; E, T)$ is the Heston price of a European vanilla call option having as underlying the stock whose log-return is described by the Heston model;
2. E and T denote, respectively, the strike price and the maturity time of the vanilla call option. Moreover we assume $T > t_n$;
3. $\tilde{v}_i$ denotes the stochastic variance at time $t = t_i$. Remind that $\tilde{v}_i$ cannot be observed;
4. u_i is sampled from a Gaussian random variable with mean zero and known variance ϕ_i.

We want to use the data available to solve the following problems:

1. Estimation Problem: find an estimate of the vector $\underline{\Theta} = (\mu, \gamma, \varepsilon, \theta, \rho, \tilde{v}_0, \lambda)^T$.
2. Filtering Problem (Forecasting Problem): given the values of the model parameters $\underline{\Theta} = (\mu, \gamma, \varepsilon, \theta, \rho, \tilde{v}_0, \lambda)^T$ find the stochastic variance and forecast the stock log-return and the stochastic variance.

That is the calibration problem consists in estimating the vector $\underline{\Theta}$ from the data given by the observations at time $t = t_i$ of the stock log-return $\tilde{x}_i$ and of the option price $\tilde{C}_i$, for $i = 0, 1, \ldots, n$, i.e. consists in estimating the value of the vector $\underline{\Theta}$ that makes most likely the available observations $\mathcal{F}_t = \{(\tilde{x}_i, \tilde{C}_i) : t_i \leq t\}$, $t > 0$.

As a byproduct of the solution of this calibration problem we obtain a technique to track the unknown stochastic variance v_t, $t > 0$, i.e. once the vector $\underline{\Theta}$ is known, we can estimate the stochastic variance v_t, $t > 0$, as mean value of the random variable v_t with respect to the probability density function conditioned to the observations associated to the Heston model corresponding to the vector $\underline{\Theta}$. The filtering problem that we consider consists in finding the probability density function conditioned to the observations associated to the Heston model (see Mariani et al. 2008).

Moreover using the probability density function found the stock log-return and the stochastic variance can be forecasted for $t > t_n$.

1.1.5 Solution of the Calibration Problem

Let

1. $p(x, v, t|\mathcal{F}_t, \underline{\Theta})$ be the joint probability density function of the random variables x_t and v_t at time $t > 0$ conditioned to the observations $\mathcal{F}_t$;
2. $p_i(x, v, t|\underline{\Theta}) = p(x, v, t|\mathcal{F}_{t_i}, \underline{\Theta})$, be the joint probability density function of the random variables x_t, v_t, conditioned to the observations made up to time $t = t_i$, $t_i < t \leq t_{i+1}$, $i = 0, 1, \ldots, n$.

In order to measure the likelihood of the vector $\underline{\Theta}$ we introduce a (log-)likelihood function:

$$F(\underline{\Theta}) = \sum_{i=0}^{n-1} \log \left[\int_0^{+\infty} p_i(\tilde{x}_{i+1}, v, t_{i+1} | \underline{\Theta}) \pi_1(\tilde{x}_{i+1}, v, t_{i+1} | \underline{\Theta}) dv \right]$$

$$+ \log \left[\int_0^{+\infty} p_0(\tilde{x}_0, v, t_0 | \underline{\Theta}) \pi_1(\tilde{x}_0, v, t_0) \right], \underline{\Theta} \in \mathcal{M},$$

where

$$\pi_1(\tilde{x}_i, v, t_i | \underline{\Theta}) = \frac{1}{\sqrt{2\pi\phi_i}} \exp\left(-\frac{1}{2\phi_i} (\tilde{C}_i - C(\tilde{x}_i, v, t_i; E, T, \underline{\Theta}))^2 \right),$$

$$(\tilde{x}_i, v) \in (-\infty, +\infty) \times (0, +\infty), i = 0, 1, \ldots, n.$$

The solution of the estimation problem is given by the vector $\underline{\Theta}$ that solves the following optimization problem:

$$\max_{\underline{\Theta} \in \mathcal{M}} F(\underline{\Theta}). \tag{1}$$

This problem is called maximum likelihood problem and is an optimization problem with nonlinear objective function and nonlinear constraints. In order to solve problem (1), we must evaluate the (log-) likelihood function $F(\underline{\Theta})$, i.e. we must find the joint probability density functions: $p_i(x, v, t | \underline{\Theta})$, $(x, v) \in (-\infty, +\infty) \times (0, +\infty)$, $t_i < t \leq t_{i+1}$, $\underline{\Theta} \in \mathcal{M}$, for $i = 0, 1, ..., n$.

The probability density functions p_i, $i = 0, 1, ..., n-1$, are solutions of the following Fokker-Planck equation associated to the Heston model: for $i = 0, 1, ..., n-1$,

$$\begin{cases} \frac{\partial p_i}{\partial t} = \frac{1}{2}\left[\frac{\partial^2(vp_i)}{\partial x^2} + \frac{\partial^2(\varepsilon^2 vp_i)}{\partial v^2} + 2\frac{\partial^2(\varepsilon\rho vp_i)}{\partial x\partial v}\right] \\ \qquad -\frac{\partial((\mu-\frac{1}{2}v)p_i)}{\partial x} - \frac{\partial(\gamma(\theta-v)p_i)}{\partial v}, \\ (x,v) \in (-\infty,+\infty)\times(0,+\infty), \quad t_i < t \le t_{i+1}, \\ p_i(x,v,t_i|\underline{\Theta}) = f_i(x,v;\underline{\Theta}), \quad (x,v)\in(-\infty,+\infty)\times(0,+\infty), \end{cases}$$

where

$$f_0(x,v;\underline{\Theta}) = \delta(x-\tilde{x}_0)\delta(v-\tilde{v}_0), (x,v)\in(-\infty,+\infty)\times(0,+\infty),$$

$$f_i(x,v;\underline{\Theta}) \quad = \quad \frac{\delta(x-\tilde{x}_i)p_{i-1}(x,v,t_i|\underline{\Theta})\pi_1(x,v,t_i|\underline{\Theta})}{\int_0^{+\infty} p_{i-1}(\tilde{x}_i,v',t_i|\underline{\Theta})\pi_1(\tilde{x}_i,v',t_i|\underline{\Theta})dv'},$$

$$(x,v)\in(-\infty,+\infty)\times(0,+\infty), \quad i=1,2,...,n.$$

Remark

The probability density functions p_i, $i = 0,1,...,n-1$, solutions of the Fokker-Planck equation can be obtained as an integral with respect to the state variables of the product of the fundamental solution of the Fokker-Planck equation associated to the Heston model with the initial condition.

1.1.6 The Filtering Problem

Let us assume that the vector $\underline{\Theta}$ and the filtration $\mathcal{F}_t = \{(\tilde{x}_i, \tilde{C}_i) : t_i \le t\}$ are given.

From the knowledge of the joint probability density function $p(x, v, t|\mathcal{F}_t, \underline{\Theta})$, $(x, v) \in (-\infty, +\infty) \times (0, +\infty)$, $t \geq 0$, we can forecast the values of the stock log-return x_t, $t > 0$, $t \neq t_i$, $i = 1, 2, \ldots, n$, and of the stochastic variance v_t, $t > 0$, using respectively the mean values $\hat{x}_{t|\underline{\Theta}}$, $\hat{v}_{t|\underline{\Theta}}$, $t > 0$, conditioned to the observations of the random variables x_t, v_t, $t > 0$, i.e. the <u>forecasted values</u> $\hat{x}_{t|\underline{\Theta}}$, $\hat{v}_{t|\underline{\Theta}}$ of x_t, v_t, $t > 0$, are given by:

$$\hat{x}_{t|\underline{\Theta}} = \mathbb{E}(x_t|\mathcal{F}_t, \underline{\Theta}) = \int_0^{+\infty} dv \int_{-\infty}^{+\infty} dx\, x\, p(x, v, t|\mathcal{F}_t, \underline{\Theta}), t > 0,$$

$$\hat{v}_{t|\underline{\Theta}} = \mathbb{E}(v_t|\mathcal{F}_t, \underline{\Theta}) = \int_0^{+\infty} dv \int_{-\infty}^{+\infty} dx\, v\, p(x, v, t|\mathcal{F}_t, \underline{\Theta}), t > 0.$$

A 1D integral representation formula of the option price.

Let r be the risk free interest rate, then we find that the price of a call option with strike price E and maturity time T is given by the following 1D Fourier integral (see Fatone et al. 2008 and Mariani et al. 2008):

$$C(x', v', t'; E, T, \underline{\Theta}) = \frac{e^{-r(T-t')}}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{rT(2-ik)} e^{(ik-1)\log E}}{2 + 3ik - k^2}$$

$$e^{-\frac{2\gamma^*\theta^*}{\varepsilon^2}\left(\log \frac{\beta^*}{2\zeta^*} + (\xi^* + \zeta^*)(T-t')\right)} e^{-\frac{2v'\alpha^*}{\varepsilon^2 \chi} + \tilde{M}^* \tilde{v}^* + x'(2-ik)} dk,$$

$$(x', v') \in (-\infty, +\infty) \times (0, +\infty), \quad t' > 0,$$

where

$$\gamma^* = \gamma + \lambda, \qquad \theta^* = \gamma\theta/(\gamma + \lambda),$$

$$\xi^* = \xi^*(k) = -\frac{1}{2}(\varepsilon\rho ik - 2\rho\varepsilon + \gamma + \lambda),$$

$$\zeta^* = \zeta^*(k) = \frac{1}{2}(4\xi^{*^2}(k) + \varepsilon^2(k^2 + 3ik - 2))^{1/2}$$

and $\alpha^* = \alpha^*(k,\tau)$, $\beta^* = \beta^*(k,\tau)$, $\chi^*(k,\tau)$ are given by:

$$\alpha^* = \alpha^*(k,\tau) = \xi^*(k) + \zeta^*(k) + (-\xi^*(k) + \zeta^*(k))e^{-2\zeta^*(k)\tau},$$

$$\beta^* = \beta^*(k,\tau) = -\xi^*(k) + \zeta^*(k) + (\xi^*(k) + \zeta^*(k))e^{-2\zeta^*(k)\tau},$$

$$\chi^* = \chi^*(k,\tau) = 1 - e^{-2\zeta^*(k)\tau},$$

$$\tilde{M}^*(k,\tau) = 2\beta^*(k,\tau)/(\varepsilon^2\chi^*(k,\tau)),$$

$$\tilde{v} = 4\zeta^{*^2}(k)e^{-2\zeta^*(k)\tau}v'/\beta^{*^2}(k,\tau), v' \in (0,+\infty).$$

Remark

This 1D integral can be computed easily by numerical quadrature. Note that in comparison to previous work of the same authors this new 1D integral representation formula simplifies substantially the numerical evaluation of the initial conditions of the filtering problem and as a consequence the evaluation of the (log-)likelihood function (see Fatone et al. 2008 and Mariani et al. 2008).

The iterative algorithm used in the maximum likelihood problem

The technique used to solve the maximum likelihood problem is based on a variable metric steepes ascent method (see Herzel et al. 1991).

Beginning from an initial guess $\underline{\Theta}^0$, we update at every iteration the current approximation of the solution of the optimization problem with a step in the direction of the gradient of the (log-)likelihood function computed in a suitable metric.

Note that $\underline{\Theta}^0 \in \mathcal{M}$ is built with some elementary ad hoc steps.

Let us fix a tolerance value $\delta > 0$ and a maximum number of iterations $iter > 0$, we denote with $\underline{\Theta}^*$ the maximizer of the (log-)likelihood function.

1. Set $k = 0$ and initialize $\underline{\Theta} = \tilde{\underline{\Theta}}^0$;

2. Evaluate $F(\underline{\Theta}^k)$, if $k > 0$ and if $|F(\underline{\Theta}^k) - F(\underline{\Theta}^{k-1})| < \delta$, where $|\cdot|$ denotes the absolute value of $\cdot$, go to item 7;

3. Evaluate the gradient of the (log-)likelihood function: $\nabla F(\underline{\Theta}^k) = \left(\frac{\partial F}{\partial \mu}, \frac{\partial F}{\partial \gamma}, \frac{\partial F}{\partial \varepsilon}, \frac{\partial F}{\partial \theta}, \frac{\partial F}{\partial \rho}, \frac{\partial F}{\partial \tilde{v}_0}, \frac{\partial F}{\partial \lambda}\right)^T (\underline{\Theta}^k)$, if $\|\nabla F(\underline{\Theta}^k)\| < \delta$ where $||\cdot||$ denotes the Euclidean norm of the vector $\cdot$, go to item 7;

4. Perform the steepest ascent step, evaluating $\underline{\Theta}^{k+1} = \underline{\Theta}^k + \eta_k \nabla F(\underline{\Theta}^k)$, where η_k is a positive real number representing the length of the step done in the direction of $\nabla F(\underline{\Theta}^k)$. The choice of η_k involves the use of “variable metrics”;

5. If $||\underline{\Theta}^{k+1} - \underline{\Theta}^k|| < \delta$, go to item 7;

6. Set $k = k + 1$, if $k < iter$ go to item 2;

7. Set $\underline{\Theta}^* = \underline{\Theta}^k$ and stop.

1.1.7 Some Numerical Results

Example 1: Calibration of the Heston model using synthetic data

We consider a "year" made of 252.5 trading days, a temporal interval of 36 consecutive trading days and $n = 9$ observation times in this time period.

We choose $t_0 = 0$, and $t_i = t_{i-1} + \Delta$, $i = 1, 2, ..., 9$, where $\Delta = \frac{4}{252.5}$.

The vector $\underline{\Theta}$ used to generate one trajectory of the Heston model is:

$$\underline{\Theta} = (\mu, \gamma, \varepsilon, \theta, \rho, \tilde{v}_0, \lambda_0)^T = \tilde{\underline{\Theta}}$$

$$= (0.026, 5.94, 0.306, 0.01159, -0.576, 0.5, 0)^T.$$

We have $F(\tilde{\underline{\Theta}}) \cong 43.15$.

Starting from $\underline{\Theta}^0 = (0.5, 8.22, 0.15, 0.0067, -0.634, 0.3, 0.01) \in \mathcal{M}$, with $\|\tilde{\underline{\Theta}} - \underline{\Theta}^0\| \approx 2.3$, to evaluate the function $F(\underline{\Theta})$ and its gradient using finite differences we solve eight filtering problems for each choice of $\underline{\Theta}$ in the optimization procedure.

The synthetic data obtained integrating numerically with Euler method one trajectory of the Heston model and evaluating the option prices with the formula (1D integral) shown above are the following:

i	t_i(year)	$\tilde{x}_i$	$\tilde{C}_i$
0	0	0	0.5099
1	$4/252.5$	1.2355e-1	0.6410
2	$8/252.5$	6.8935e-2	0.5804
3	$12/252.5$	-2.9809e-2	0.4791
4	$16/252.5$	1.2049e-2	0.5201
5	$20/252.5$	4.4983e-2	0.5538
6	$24/252.5$	-7.7662e-4	0.5065
7	$28/252.5$	-1.3411e-2	0.4936
8	$32/252.5$	-1.2461e-1	0.3894
9	$36/252.5$	-7.1041e-2	0.4377

Note that we have chosen $\phi_i = 0.0005$, $i = 0, 1, \ldots, n$.

To each vector $\underline{\Theta}^k$=$(\Theta_1^k, \Theta_2^k, \ldots, \Theta_7^k)^T$, $k = 1, 2, ..., iter$, generated by the optimization procedure there is associated a function $\tilde{p}(x, v, t|\mathcal{F}_t, \underline{\Theta}^k)$, $(x, v) \in (-\infty, +\infty) \times (0, +\infty)$ $t > 0$, solution of a corresponding filtering problem, with initial stochastic variance $\tilde{v}_0 = \Theta_6^k$, risk neutral parameter $\lambda = \Theta_7^k$ and parameter values Θ_j^k, $j = 1, 2, ..., 5$.

So that for each vector $\underline{\Theta}^k$ we can estimate the stock log-returns and the stochastic variance values using the correspondent probability density function $\tilde{p}(x, v, t|\mathcal{F}_t, \underline{\Theta}^k)$, $(x, v) \in (-\infty, +\infty) \times (0, +\infty)$, $t > 0$.

We fix $iter = 200$ and we use the variable metric steepest ascent method with the following stopping criterion:

$$|F(\underline{\Theta}^k) - F(\underline{\Theta}^{k-1})| \leq 5.e - 6, \text{ and k} > 10.$$

Note that the maximum likelihood procedure finds as estimate of the vector $\underline{\Theta}$ a vector $\underline{\Theta}^* = \underline{\Theta}^{92}$, such that $F(\underline{\Theta}^*) = F(\underline{\Theta}^{92}) > F(\tilde{\underline{\Theta}})$.

solid line: "true trajectory", dash-dotted line: "forecasted trajectory"

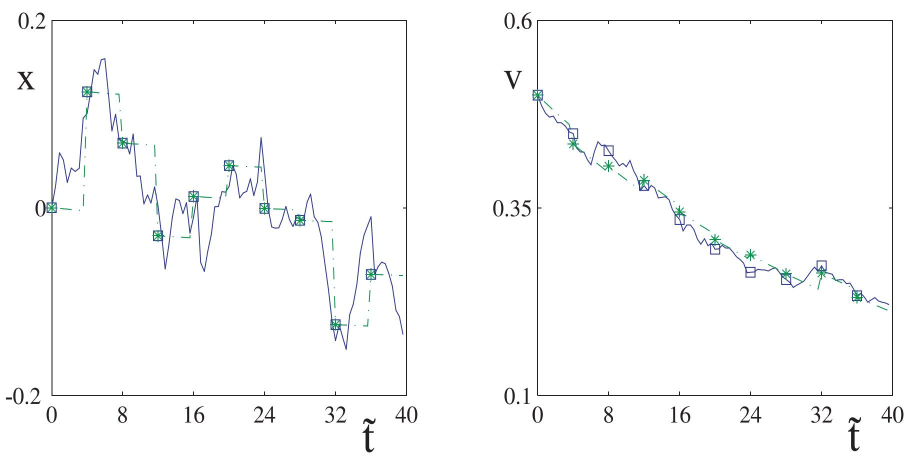


Figure 1. *Tracking of the trajectory and quality of the forecasts when* $\underline{\Theta} = \tilde{\underline{\Theta}}$

Example 2: Analysis of $S\&P500$ index

We analyze the historical data of the $S\&P500$ index in the year 2005. We consider the daily closing values of the $S\&P500$ index and the closing prices of a European call option on the $S\&P500$ index with maturity date December 16, 2005 and strike value $E = 1200$ during the period of about six months from January 3, 2005 to March 3, 2005.

We choose as time $t = 0$ the first day January 3, 2005 and we consider 10 observation times, $t_i = 4\,i/253$, $i = 0, 1, \ldots, 9$. We apply the procedure

described with $\delta = 5.e - 07$ and the starting point has been chosen taking into account the implied volatility of the option, the standard deviation and the mean value of the historical log-returns data. The calibration procedure gives the following parameter vector $\underline{\Theta} = \underline{\Theta}^c = (\mu, \gamma, \varepsilon, \theta, \rho, \tilde{v}_0, \lambda)^T$
$= (0.012, 4.303, 0.114, 0.0067, 0.635, 0.065, 0.00058)^T$.

solid line: "true trajectory", dash-dotted line: "forecasted trajectory"

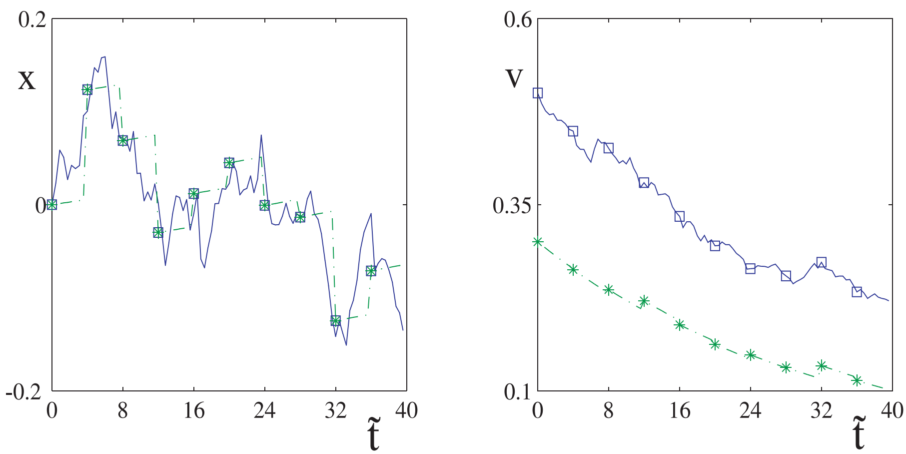


Figure 2. *Tracking of the trajectory and quality of the forecasts when* $\underline{\Theta} = \underline{\Theta}^0$

solid line: “true trajectory”, dash-dotted line: “forecasted trajectory”

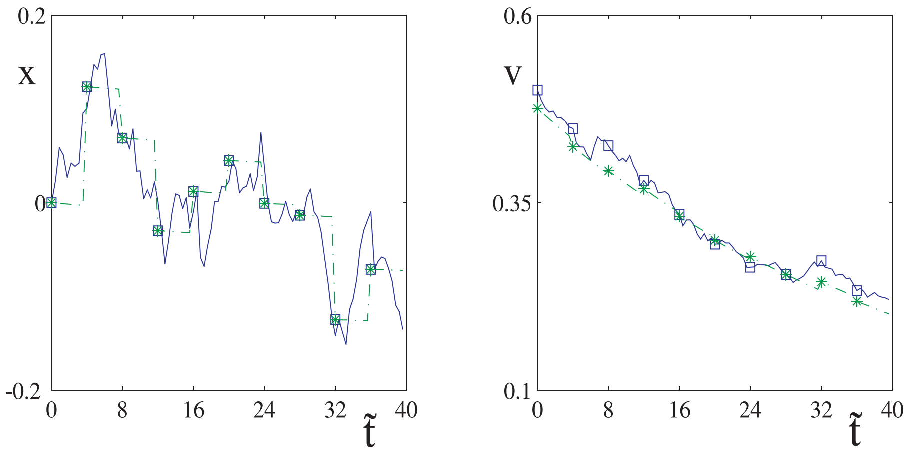


Figure 3. *Tracking of the trajectory and quality of the forecasts when* $\underline{\Theta} = \underline{\Theta}^{92} = \underline{\Theta}^{*}$

solid line: “true trajectory”, dash-dotted line: “forecasted trajectory”

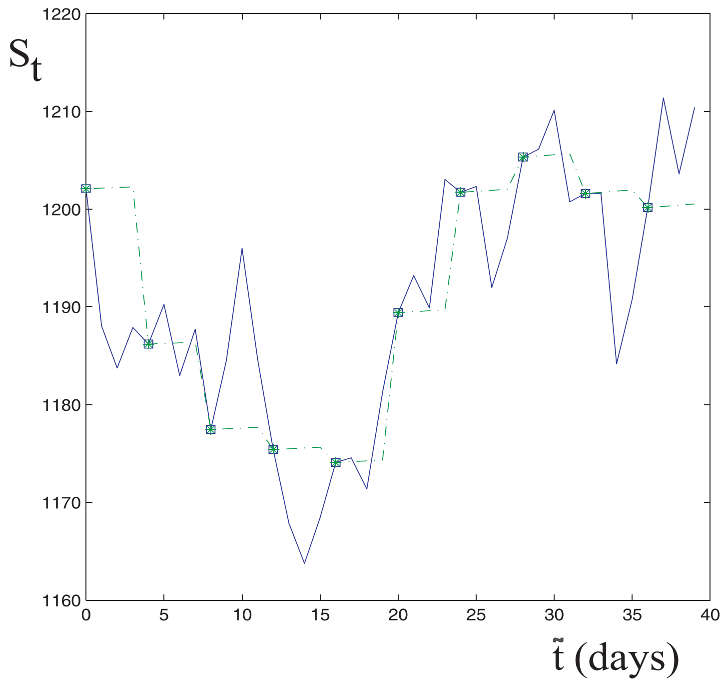


Figure 4. *Tracking of the log-returns when* $\underline{\Theta} = \underline{\Theta}^c$

solid line: “true trajectory”, dash-dotted line: “forecasted trajectory”

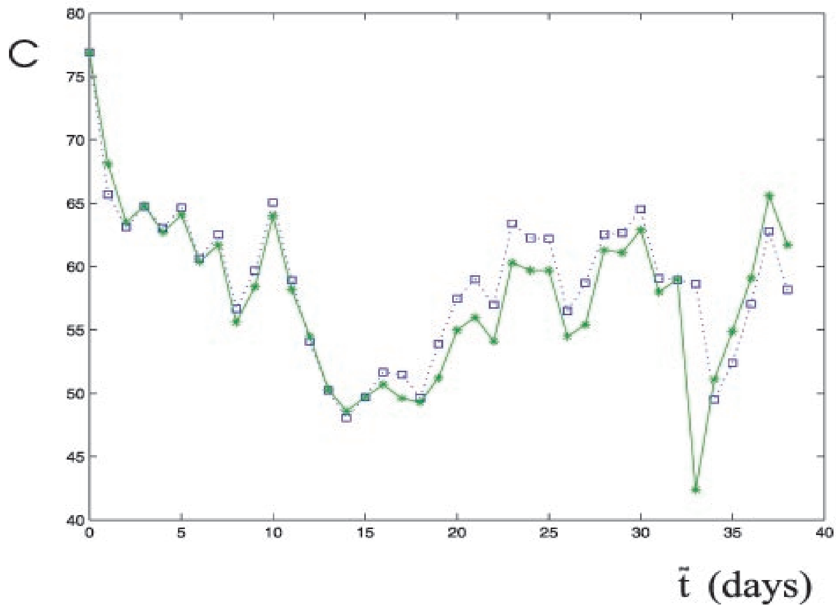


Figure 5. *Tracking of the option prices* $\underline{\Theta} = \underline{\Theta}^c$

1.1.8 References

[1] Fatone L., Mariani F., Recchioni M.C., Zirilli F. (2008). The calibration of the Heston stochastic volatility model using filtering and maximum likelihood methods, in Proceedings of Dynamic Systems and Applications, G.S.Ladde, N.G.Medhin, Chuang Peng, M.Sambandham Editors, Dynamic Publishers, Atlanta, USA, 5, 170-181.

[2] Herzel S., Recchioni M.C., Zirilli F. (1991). A quadratically convergent method for linear programming, Linear Algebra and its Applications 152, 255-289.

[3] Heston S. (1993). A closed form solution for options with volatility with applications to bond and currency options, Review of Financial Studies 6, 327-343.

[4] Mariani F., Pacelli G., Zirilli F. (2008). Maximum Likelihood Estimation of the Heston Stochastic Volatility Model Using Asset and Option Prices: an Application of Nonlinear Filtering Theory, Optimization Letters 2, 177-222.

Section 1.2

Multiscale Heston Model: Option Pricing Formulae and Calibration Problems

[Description] *We present an explicitly solvable multiscale stochastic volatility model that generalizes the Heston model. The model describes the dynamics of an asset price and of its two stochastic variances using a system of three Ito stochastic differential equations. The two stochastic variances vary on two different time scales and can be seen as auxiliary variables introduced to model the dynamics of the asset price. Under some assumptions on the correlation structure the transition probability density function of the stochastic process solution of the model the option pricing formulae are represented as one dimensional integrals of explicitly known integrands. In this sense the model is explicitly solvable. The option pricing formulae obtained are used to study the values of the model parameters, of the correlation coefficients of the Wiener processes defining the model and of the initial stochastic variances implied by the "observed" option prices using both synthetic and real data (S& P 500 index in the year 2005). The real data analysis presented shows that the multiscale stochastic volatility model can be used to obtain high quality forecasts of option prices.*

[Paper] *Fatone L., Mariani F., Recchioni M.C., Zirilli F. (2009). An explicitly solvable multi-scale stochastic volatility model: option pricing and calibration problems, Journal of Futures Markets 29(9), 862-893.*

[Website] *http://www.econ.univpm.it/recchioni/finance/w7*

1.2.1 Outline of the Presentation

- The calibration and filtering problems for a stochastic volatility model.
- The Heston model.
- The multiscale stochastic volatility model.
- The advantages of using the multiscale stochastic volatility model.
- Transition probability density function of the multiscale stochastic volatility model.
- Option pricing in the multiscale stochastic volatility model. An integral representation formula.
- Numerical experiments with synthetic and real data.
- References.

1.2.2 Calibration and Filtering Problems for a Stochastic Volatility Model

Let us consider a stochastic volatility model characterized by a vector $\underline{\Theta}$ of parameters (having one or more factors) that describes the dynamics of the stock log return x_t, $t > 0$.

We want to estimate the parameters $\underline{\Theta}$ of the stochastic volatility model starting from price data.

We use as data the observation at discrete times of the stock log-returns and/or of the prices of European vanilla call/put options on the stock. The approach that we propose makes use of:

- nonlinear filtering techniques,
- maximum likelihood and mean least squares methods.

The problem considered is realistic and the analysis of time series of real financial data has been carried out. The results obtained are very satisfactory (Fatone et al. 2008, 2009).

We want to solve the following problems:

1) Estimation Problem: find an estimate of the vector $\underline{\Theta}$ from the observations, that is, for example, from the knowledge at time $t = t_i$ ($t_i < t_{i+1}, t_{n+1} = +\infty$) of the stock log-return $\tilde{x}_i$ and/or of an option price $\tilde{C}_i$, for $i = 0, 1, \ldots, n$. This means find the value of the vector $\underline{\Theta}$ that makes most likely the available observations $\mathcal{F}_t = \{(\tilde{x}_i, \tilde{C}_i) : t_i \leq t\}, t > t_0$.

2) Filtering Problem (Forecasting Problem): given the values of the model parameters $\underline{\Theta}$ forecast the stock log-return for $t \neq t_i$, $i = 1, 2, \ldots, n$ and the stochastic variance (or variances) for $t > t_0$, and in particular for $t > t_n$.

Note that in the solution of these two problems we use the joint transition probability density function of x_t and of the associated stochastic variances, a formula to price an option under this stochastic model and the joint probability density function of x_t and of the associated stochastic variances conditioned to the observations.

We make the forecasts of the state variables of the model taking the mean values of the variables with respect to joint probability density functions conditioned to the observations.

1.2.3 Heston Stochastic Volatility Model

Let S_t, $t > 0$, and x_t, $t > 0$, be the stock price and the stock log-return at time t ($x_t = \log S_t/S_0$), v_t, $t > 0$, be the stochastic variance associated to the stock log-return x_t at time t.

One of the most important stochastic volatility model is the <u>Heston model</u> (Heston 1993) that describes the dynamics of x_t, $t > 0$, and of v_t, $t > 0$, as follows:

$$\begin{aligned} dx_t &= \left(\mu - \frac{1}{2}v_t\right)dt + \sqrt{v_t}\, dW_t^1, \quad t > 0, \\ dv_t &= \gamma(\theta - v_t)dt + \varepsilon\sqrt{v_t}\, dZ_t^1, \quad t > 0, \end{aligned}$$

with initial conditions $x_0 = \tilde{x}_0$, $v_0 = \tilde{v}_0$, where μ, γ, ε, θ are real constants, W_t^1, Z_t^1, $t > 0$, are standard Wiener processes such that $W_0^1 = Z_0^1 = 0$, dW_t^1, dZ_t^1 are their stochastic differentials, $< dW_t^1 dZ_t^1 >= \rho dt$, where $< \cdot >$ denotes the expected value of $\cdot$, and $\rho \in [-1, 1]$ is a constant known as correlation coefficient.

1.2.4 Heston Stochastic Volatility Model (More)

$$\begin{aligned} dx_t &= \left(\mu - \frac{1}{2}v_t\right)dt + \sqrt{v_t}\, dW_t^1, \quad t > 0, \\ dv_t &= \gamma(\theta - v_t)dt + \varepsilon\sqrt{v_t}\, dZ_t^1, \quad t > 0, \end{aligned}$$

- μ drift
- γ speed of mean reverting process

- θ long term mean variance
- ϵ volatility of volatility

This model, thank to the stochastic variance v_t, improves the well known Black Scholes model and captures (to some extent) the volatility structure. The set of parameters to be estimated (using price data) is given by:

$$\underline{\Theta} = (\mu, \gamma, \varepsilon, \theta, \rho, \tilde{v}_0, \lambda)^T,$$

where $\tilde{v}_0$ is the initial stochastic variance and λ is the risk premium parameter.

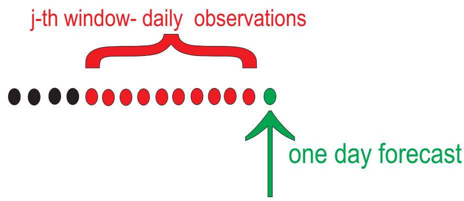
1.2.5 Heston Model Calibration: Numerical Results on Real Data

We analyze the daily closing values of the U.S. S&P 500 index and the corresponding bid prices of a European call option on the U.S. S&P 500 index with maturity date December 16, 2005 and strike price $K = 1200$ during the period of about four months going from January 3, 2005 to May 11, 2005. Due to the number of data actually available in 2005 the time unit is a year made of 253 trading days.

The calibration procedure employed uses filtering and maximum likelihood and has been introduced in Mariani et al. 2008 and further developed in Fatone et al. 2008, 2009.

We begin choosing as time $t = t_0 = 0$ the day January 3, 2005 and we have considered as data of the calibration problem 81 windows of 10 consecutive daily observation times.

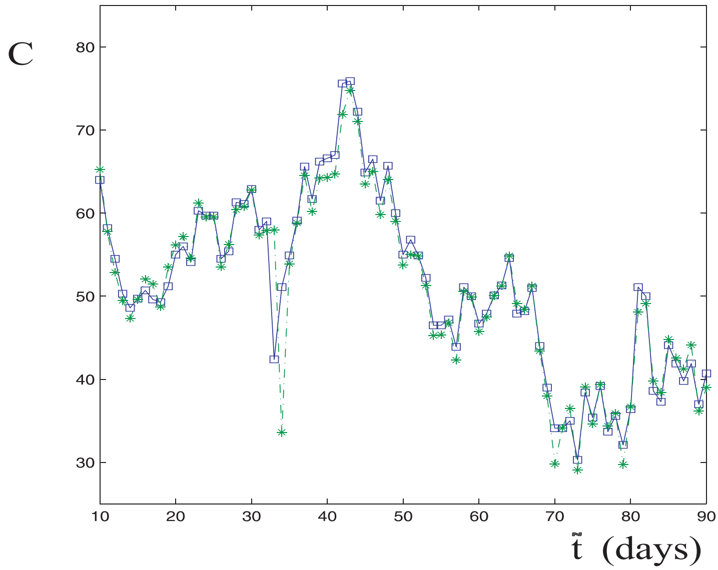
The calibration procedure, applied to the eighty-one data windows considered provides eighty-one estimated parameter vectors. We use the parameter vector estimated using the data contained in the j-th window to forecast the value of the S&P500 index and the bid price of the option the day next to the last observation time of the j-th window.



1.2.6 Heston Model Calibration: Numerical Results on Real Data (More)

We note that in the eighty-one calibration and forecasting problems solved the mean relative error made in the forecasts of the next day S&P500 index value and option bid price are 0.0063 and 0.0785 respectively.

The following figure and digital movie show the one day in the future forecasted values of the S&P 500 option bid price as function of time. The blue boxes are the historical values, the green/red stars are the one day in the future forecasted values.



Forecasting of option prices *Movie* 1

1.2.7 Heston Model: Conclusion

The results obtained using the filtering and maximum likelihood approach suggested in Mariani et al. 2008 on the Heston stochastic volatility model are particularly satisfactory when:

1. the observed stock prices present no spikes,
2. the European vanilla option prices used as data refer to at the money options,
3. the time to maturity of the options considered is not too large.

These conditions are not always satisfied.

Moreover time series analysis of financial data has shown that there exist financial assets whose prices are affected by volatility of two different time scales: a short time scale and a long time scale and that there exist prices, such as commodity prices, where the spikes are natural.

Can we improve the Heston model to take care of these facts?

1.2.8 The Multiscale Stochastic Volatility Model

$$dx_t = (\mu + a_1 v_{1,t} + a_2 v_{2,t})dt + b_1\sqrt{v_{1,t}}dW_t^1 + b_2\sqrt{v_{2,t}}dW_t^2, t > 0,$$

$$dv_{1,t} = \chi_1(\theta_1 - v_{1,t})dt + \varepsilon_1\sqrt{v_{1,t}}dZ_t^1, t > 0,$$

$$dv_{2,t} = \chi_2(\theta_2 - v_{2,t})dt + \varepsilon_2\sqrt{v_{2,t}}dZ_t^2, t > 0,$$

$$x_0 = \tilde{x}_0, v_{1,0} = \tilde{v}_{1,0}, v_{2,0} = \tilde{v}_{2,0},$$

where the quantities a_i, b_i, χ_i, ε_i, θ_i, $i = 1, 2$, are real constants and W_t^1, W_t^2, Z_t^1, Z_t^2, $t > 0$, are standard Wiener processes such that $W_0^1 = W_0^2 = Z_0^1 = Z_0^2 = 0$, dW_t^1, dW_t^2, dZ_t^1, dZ_t^2, $t > 0$, are their stochastic differentials.

Correlation structure of the model:

$$\left(\begin{array}{c|cccc}
 & W^1 & Z^1 & W^2 & Z^2 \\
\hline
 & & & & \\
W^1 & 1 & \rho_1 & 0 & 0 \\
Z^1 & \rho_1 & 1 & 0 & 0 \\
W^2 & 0 & 0 & 1 & \rho_2 \\
Z^2 & 0 & 0 & \rho_2 & 1
\end{array}\right)$$

$$dx_t = (\mu + a_1 v_{1,t} + a_2 v_{2,t})dt + b_1\sqrt{v_{1,t}}dW_t^1 + b_2\sqrt{v_{2,t}}dW_t^2, t > 0,$$

$$dv_{1,t} = \chi_1(\theta_1 - v_{1,t})dt + \varepsilon_1\sqrt{v_{1,t}}dZ_t^1, t > 0,$$

$$dv_{2,t} = \chi_2(\theta_2 - v_{2,t})dt + \varepsilon_2\sqrt{v_{2,t}}dZ_t^2, t > 0,$$

$$x_0 = \tilde{x}_0, v_{1,0} = \tilde{v}_{1,0}, v_{2,0} = \tilde{v}_{2,0},$$

The parameter vector that must be estimated is:

$$\underline{\Theta} = (\mu, \chi_1, \theta_1, \varepsilon_1, \tilde{v}_{1,0}, \lambda_1, \chi_2, \theta_2, \varepsilon_2, \lambda_2, \tilde{v}_{2,0}, \rho_1, \rho_2),$$

where λ_i, $i = 1, 2$ are the risk premium parameters.

Eventually also the parameters a_i, b_i, $i = 1, 2$ could be estimated, but, for the moment, we choose $a_1 = a_2 = -1/2$, $b_1 = b_2 = 1$. This choice is done in analogy with the Heston model.

This model has been introduced by Fatone, Mariani, Recchioni, Zirilli 2009.

1.2.9 Why Using a Multiscale Model?

1. Several empirical studies of real data have shown that the term structure of the implied volatility of the price of many underlyings seems to be driven by two different factors ($\chi_1 << \chi_2$).

2. The model is able to reproduce spikes through the use of a fast time scale volatility together with an intermediate time scale volatility. This ability can be applied to the study of commodity prices and eventually changing appropriately the meaning of the variables can

be used to the study of prices financial products that live for long time periods (such as life insurance contracts).

3. The model contains as special cases some well known models such as the Black Scholes model and the Heston model.

4. The model is explicitly solvable that is, under some assumptions, the transition probability density function of the stochastic process solution of the model is represented as a one dimensional integral of an explicitly known integrand. This property makes possible to price put and call options in the model computing one dimensional integrals.

1.2.10 The Multiscale Stochastic Volatility Model

We consider the parameter choice $a_1 = a_2 = -\frac{1}{2}$, $b_1 = b_2 = 1$, that is:

$$dx_t = (\mu - \frac{1}{2}v_{1,t} - \frac{1}{2}v_{2,t})dt + \sqrt{v_{1,t}}dW_t^1 + \sqrt{v_{2,t}}dW_t^2, t > 0,$$

$$dv_{1,t} = \chi_1(\theta_1 - v_{1,t})dt + \varepsilon_1\sqrt{v_{1,t}}dZ_t^1, t > 0,$$

$$dv_{2,t} = \chi_2(\theta_2 - v_{2,t})dt + \varepsilon_2\sqrt{v_{2,t}}dZ_t^2, t > 0,$$

$$x_0 = \tilde{x}_0, \quad v_{1,0} = \tilde{v}_{1,0}, v_{2,0} = \tilde{v}_{2,0},$$

We call this choice Double Heston model.

The model is multiscale when $0 < \chi_1 << \chi_2$.

The two stochastic variances $v_{1,t}$, $v_{2,t}$, $t > 0$, capture respectively the long term variance (slow time scale) and the short term variance (fast time scale).

Note that when we work on the calibration problem only with option prices as data we can incorporate the risk premium parameters λ_i, $i = 1, 2$, associated to the risk neutral measure into the parameters χ_i and θ_i, $i = 1, 2$.

1.2.11 Spikes Using Multiscale Stochastic Volatility Model

Multiscale trajectory parameters: $\mu = 0.03$, $\theta_1 = 0.01$, $\theta_2 = 0.03$, $\chi_1 = 1$, $\chi_2 = 100$, $\rho_1 = -0.5$, $\rho_2 = -0.7$, $\varepsilon_1 = 0.25\sqrt{\chi_1}$, $\varepsilon_2 = 2\sqrt{\chi_2}$, $\tilde{v}_{1,0} = 0.05$, $\tilde{v}_{2,0} = 0.015$;

Heston trajectory parameters: $\mu = 0.03$, $\theta_1 = 0.01$, $\chi_1 = 1$, $\rho_1 = -0.5$, $\varepsilon_1 = 0.25\sqrt{\chi_1}$, $\tilde{v}_{1,0} = 0.05$.

The choice $\chi_1 = 1$ and $\chi_2 = 100$ guarantees that the stochastic variances change on different time scales.

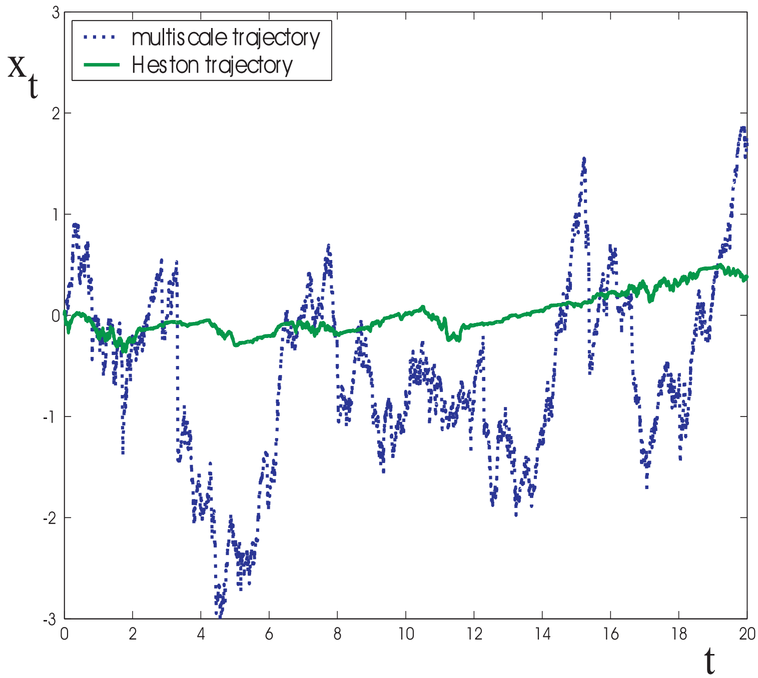


Figure 6. *Example of synthetic data*

1.2.12 Transition Probability Density Function

Let $p_f(x, v_1, v_2, t, x', v_1', v_2', t')$, $t > t'$, be the transition probability density function of the multiscale stochastic volatility model, the function p_f as a function of the "past" variables (x', v_1', v_2', t') satisfies the backward:

$$-\frac{\partial p_f}{\partial t'} = \frac{1}{2}(b_1^2 v_1' + b_2^2 v_2')\frac{\partial^2 p_f}{\partial x'^2} + \frac{1}{2}\varepsilon_1^2 v_1' \frac{\partial^2 p_f}{\partial v_1'^2} + \frac{1}{2}\varepsilon_2^2 v_2' \frac{\partial^2 p_f}{\partial v_2'^2}$$

$$+\varepsilon_1 b_1 \rho_1 v_1' \frac{\partial^2 p_f}{\partial x' \partial v_1'} + \varepsilon_2 b_2 \rho_2 v_2' \frac{\partial^2 p_f}{\partial x' \partial v_2'} + \chi_1(\theta_1 - v_1')\frac{\partial p_f}{\partial v_1'}$$

$$+\chi_2(\theta_2 - v_2')\frac{\partial p_f}{\partial v_2'} + (\mu + a_1 v_1' + a_2 v_2')\frac{\partial p_f}{\partial x'}$$

$$(x', v_1', v_2') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ 0 \leq t' < t,$$

with final condition:

$$p_f(x, v_1, v_2, t, x', v_1', v_2', t) = \delta(x' - x)\delta(v_1' - v_1)\delta(v_2' - v_2),$$

$$(x, v_1, v_2), (x', v_1', v_2') \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+,\ t \geq 0,$$

and the appropriate boundary conditions.

1.2.13 One Dimensional Integral Formula for Transition Probability Density Function

In order to derive a representation formula for the transition probability density function we proceed as suggested by A. Lipton (*Mathematical methods for foreign exchange*, World Scientific Pubblishing Co. Pte. Ltd, Singapore, (2001)):

1. write the backward equation satisfied by p_f as a function of the "past variables" (the equation written in the previous slide);
2. take the Fourier transform of p_f with respect to the "future variables";
3. this Fourier transform satisfies the backward equation with suitable final condition;

4. reduce the solution of the backward equation to the solution of a system of ordinary differential equations involving Riccati equations depending on some parameters;
5. find an explicit formula for the Fourier transform of p_f solving the system of ordinary differential equations, represent p_f as a three dimensional inverse Fourier transform and compute explicitly two of the three resulting one dimensional integrals.

Proceeding as described previously we obtain:

$$p_f(x, v_1, v_2, t, x', v_1', v_2', t')$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} dk\, e^{\imath k(x-x'-\mu\tau)} \cdot \prod_{i=1}^{2} e^{-2\chi_i\theta_i((\nu_i+\zeta_i)\tau+\ln(s_{i,b}/(2\zeta_i)))/\varepsilon_i^2}\cdot$$
$$\left[e^{-2v_i'(\zeta_i^2-\nu_i^2)s_{i,g}/(\varepsilon_i^2 s_{i,b})} e^{-M_i(\tilde{v}_i+v_i)} M_i \left(\frac{v_i}{\tilde{v}_i}\right)^{(\chi_i\theta_i/\varepsilon_i^2)-1/2} \cdot \right.$$
$$\left. I_{2\chi_i\theta_i/\varepsilon_i^2-1}\left(2M_i(\tilde{v}_i v_i)^{1/2}\right)\right],$$
$$(x, v_1, v_2), (x', v_1', v_2') \in \mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+, t, t' \geq 0, t-t' > 0,$$

where the quantities $s_{i,b}$, $s_{i,g}$, $\tilde{v}_i$, M_i, $i = 1, 2$ are elementary functions.

1.2.14 The Elementary Functions Defining the Transition Probability Density Function

The quantities $s_{i,b}$, $s_{i,g}$, $\tilde{v}_i$, M_i, $i = 1, 2$ are given by:

$$s_{i,g} = 1 - e^{-2\zeta_i\tau}, s_{i,b} = \zeta_i - \nu_i + (\zeta_i + \nu_i)e^{-2\zeta_i\tau}, \tau > 0, i = 1, 2,$$

$$\tilde{v}_i = \frac{4v_i'\zeta_i^2 e^{-2\zeta_i\tau}}{(s_{i,b})^2}, \; M_i = \frac{2s_{i,b}}{{\varepsilon_i}^2 s_{i,g}}, \;\; \tau > 0, \; i = 1, 2,$$

where

$$\nu_i = -\frac{1}{2}\left(\chi_i + \imath\, k\, b_i\varepsilon_i\rho_i\right), \;\; k \in \mathbb{R}, \; i = 1, 2\,,$$

$$\zeta_i = \frac{1}{2}\left(4\nu_i^2 + \varepsilon_i^2(b_i^2k^2 + 2\imath\, k\, a_i)\right)^{1/2}, \;\; k \in \mathbb{R}, \; i = 1, 2\,.$$

1.2.15 European Vanilla Call Option

Using the risk neutral formula and proceeding as done to derive the integral formula for p_f, the price of a European vanilla call option at time $t = 0$ with time to maturity $\tau > 0$, strike price E and asset price S_0 at time $t = 0$ is given:

$$C(\tau, E, S_0, \tilde{v}_{1,0}, \tilde{v}_{2,0}) = \frac{S_0}{2\pi}e^{-r\tau}e^{2\mu\tau}\int_{-\infty}^{+\infty} dk \frac{e^{-\imath k(\log(S_0/E)+\mu\tau)-\log(E/S_0)}}{-k^2 - 3\imath\, k + 2}$$

$$\prod_{i=1}^{2}\left(e^{-2\chi_i\theta_i(\nu_i^c+\zeta_i^c+\log(s_{i,b}^c/(2\zeta_i^c)))\tau/\varepsilon_i^2}e^{-2\tilde{v}_{i,0}((\zeta_i^c)^2-(\nu_i^c)^2)s_{i,g}^c/(\varepsilon_i^2 s_{i,b}^c)}\right),$$

$$\tilde{v}_{1,0}, \tilde{v}_{2,0} > 0,$$

where r is the risk free interest rate,

$$\nu_i^c = -\frac{1}{2}\left(\chi_i + \imath\, k\, b_i\varepsilon_i\rho_i - 2b_i\rho_i\varepsilon_i\right), \; k \in \mathbb{R}, \;\; i = 1, 2,$$

$$\zeta_i^c = \frac{1}{2}\left(4(\nu_i^c)^2 + \varepsilon_i^2(b_i^2k^2 + 2\imath\, k\, a_i + 4\imath\, kb_i^2 - 4(a_i + b_i^2))\right)^{1/2},$$

$$k \in \mathbb{R}, \;\; i = 1, 2$$

$$s_{i,g}^c = 1 - e^{-2\zeta_i^c\tau}, s_{i,b}^c = \zeta_i^c - \nu_i^c + (\zeta_i^c + \nu_i^c)e^{-2\zeta_i^c\tau}, \tau > 0, i = 1, 2.$$

1.2.16 European Vanilla Put Option

In a similar way using the risk neutral formula we derive the following formula for the price of a European vanilla put option at time $t = 0$ with time to maturity $\tau > 0$, strike price E and asset price S_0 at time $t = 0$:

$$P(\tau, E, S_0, \tilde{v}_{1,0}, \tilde{v}_{2,0}) = \tfrac{E}{2\pi} e^{-r\tau} e^{-\mu\tau} \int_{-\infty}^{+\infty} dk \tfrac{e^{-\imath k(\log(S_0/E)+\mu\tau) - \log(S_0/E)}}{-k^2+3\imath k+2} \cdot$$

$$\prod_{i=1}^{2} \left(e^{-2\chi_i\theta_i(\nu_i^p+\zeta_i^p+\log(s_{i,b}^p/(2\zeta_i^p)))\tau/\varepsilon_i^2} e^{-2\tilde{v}_{i,0}((\zeta_i^p)^2-(\nu_i^p)^2)s_{i,g}^p/(\varepsilon_i^2 s_{i,b}^p)} \right),$$

$$\tilde{v}_{1,0}, \tilde{v}_{2,0} > 0,$$

$$\nu_i^p = -\frac{1}{2}\left(\chi_i + \imath\, k\, b_i\varepsilon_i\rho_i + b_i\rho_i\varepsilon_i\right),\ k \in \mathbb{R},\ i = 1, 2,$$

$$\zeta_i^p = \frac{1}{2}\left(4(\nu_i^p)^2 + \varepsilon_i^2(b_i^2k^2 + 2\imath\, k\, a_i - 2\imath\, k b_i^2 - 2(a_i + b_i^2))\right)^{1/2}, k \in \mathbb{R}, i = 1, 2$$

$$s_{i,g}^p = 1 - e^{-2\zeta_i^p\tau},\ s_{i,b}^p = \zeta_i^p - \nu_i^p + (\zeta_i^p + \nu_i^p)e^{-2\zeta_i^p\tau},\ \ \tau > 0,\ i = 1, 2.$$

1.2.17 Calibration Problem

Let $\mathbb{R}^{11}$ be the 11 dimensional real Euclidean vector space and let $\mathcal{M}$ be the set of the admissible vectors $\underline{\Theta}$, that is:

$$\mathcal{M} = \{\underline{\Theta} = (\epsilon_1, \theta_1, \rho_1, \chi_1, \tilde{v}_{0,1}, \mu, \epsilon_2, \theta_2, \rho_2, \chi_2, \tilde{v}_{0,2}) \in \mathbb{R}^{11} \mid$$

$$\varepsilon_i, \chi_i, \theta_i \geq 0, i = 1, 2, \frac{2\chi_i\theta_i}{\varepsilon_i^2} \geq 1, -1 \leq \rho_i \leq -1, \tilde{v}_{0,i} \geq 0, i = 1, 2\},$$

at time t, $t \geq 0$, we solve the following optimization problem:

$$\min_{\underline{\Theta} \in \mathcal{M}} L_t(\underline{\Theta}),$$

where the objective function $L_t(\underline{\Theta})$, $t \geq 0$, is defined as follows:

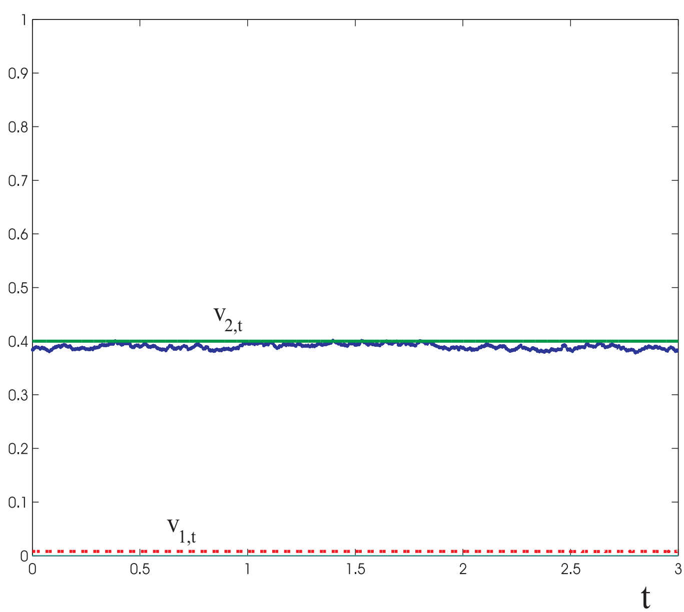
$$L_t(\underline{\Theta}) = \sum_{i=1}^{m_c} \left[C^{t,\underline{\Theta}}(\tilde{S}_t, T_i, K_i) - C^t(\tilde{S}_t, T_i, K_i) \right]^2$$

$$+ \sum_{i=1}^{m_p} \left[P^{t,\underline{\Theta}}(\tilde{S}_t, T_i, K_i) - P^t(\tilde{S}_t, T_i, K_i) \right]^2,$$

and C^t, P^t are the observed prices (data) of European vanilla call and put options respectively.

1.2.18 Numerical Results on Synthetic Data

We generate a set of synthetic data (synthetic option prices) generating the option prices with the Black Scholes formula choosing the volatility σ equal to $\sqrt{0.4}$ and the risk free interest rate r equal to 0.03.



Trajectories of the stochastic variances $v_{1,t}$ (red dotted line corresponding to a reconstructed χ_1 approximately equal to $9.96 \cdot 10^{-4}$), $v_{2,t}$ (blue dotted line corresponding to a reconstructed χ_2 approximately equal to 7.71) obtained solving the calibration problem using as data the option prices generated with the Black Scholes model and corresponding trajectory of the "true" Black Scholes variance (green line) versus time t.

1.2.19 Numerical Results on Real Data: S&P 500 Index

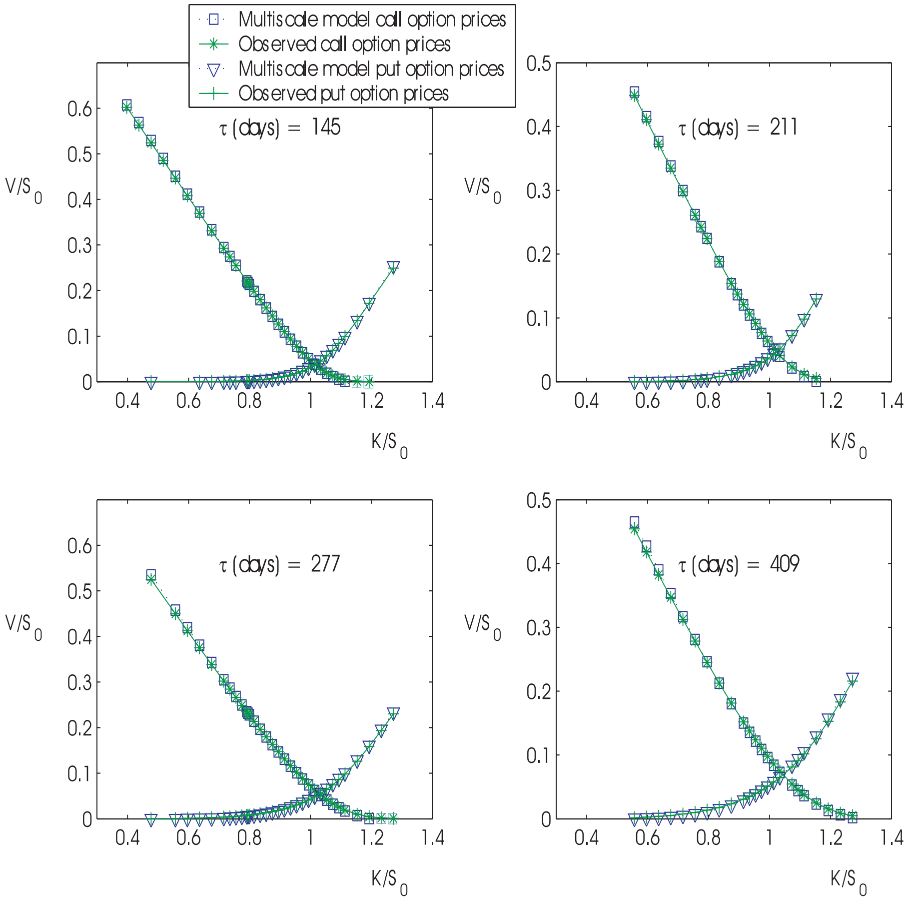
We study the values of the model parameters, of the correlation coefficients and of the initial stochastic variances implied by the observed prices of the European vanilla call and put options on the S&P 500 index and by the value of the S&P 500 index in January, June and November 2005.

For each month we proceed solving the calibration problem using all (in, at, out the money) the call and put option prices available to us relative to third day of the month, that is for example November 3, 2005 ($m_c = 303$ call options and $m_p = 284$ put options).

The implied values obtained solving the calibration problem using the data of November 3, 2005 (i.e. the third day of the month) are used to forecast the option prices of November 7 ($m_c = 303$, $m_p = 290$), November 14 ($m_c = 305$, $m_p = 295$), and November 28 ($m_c = 292$, $m_p = 265$), 2005.

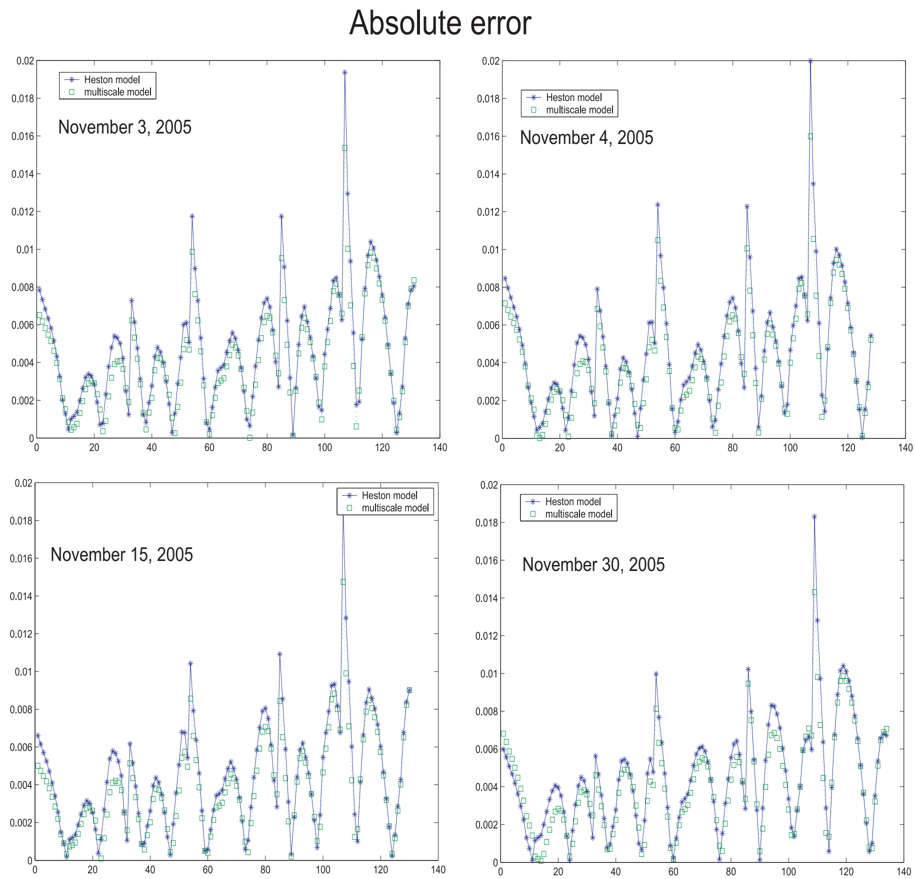
The calibration procedure works simultaneously on out of money, at the money and in the money call and put options.

1.2.20 Forecasted Values of Call and Put Options



November 28, 2005: European vanilla call and put option prices (V) on the $S\&P500$ index forecasted using the multiscale model and prices observed in the market (November 3, 2005) versus moneyness K/S_0.

1.2.21 Absolute Errors on Forecasted Values of Call Option Prices Heston-Multiscale Model



1.2.22 Future Work

- Develop a high performance parallel optimization method to solve the calibration problem.
- Extend the previous work to other kinds of financial derivatives and to the study of commodity prices.

Note that several numerical experiments and digital movies relative to the problem considered here that show the behaviour of the method proposed to solve the estimation problem can be found in the website: http://www.econ.univpm.it/recchioni/finance/w8.

A more general reference to the work in mathematical finance of the authors and of their coauthors is the website:
http://www.econ.univpm.it/recchioni/finance.

1.2.23 References

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Section 1.3

SABR and Multiscale SABR Models: Option Pricing and Calibration

[Description] *A multiscale SABR model that describes the dynamics of forward prices/rates is presented. New closed form formulae for the transition probability density functions of the normal and lognormal SABR and multiscale SABR models and for the prices of the corresponding European call and put options are deduced. The technique used to obtain these formulae is rather general and can be used to study other stochastic volatility models. A calibration problem for the models considered is formulated and solved. Numerical experiments with real data are presented.*

[Paper] *Fatone L., Mariani F., Recchioni M.C., Zirilli F. (2013). Some explicitly solvable SABR and multiscale SABR models: option pricing and calibration, Journal of Mathematical Finance 3, 10-32.*

[Website] *http://www.econ.univpm.it/recchioni/finance/w14*

1.3.1 Outline of the Presentation

- Notations
- Motivations and background
- SABR and multiscale SABR models
- Hull and White model
- Kernels of the SABR models and of the Hull and White model
- Option pricing formulae in the SABR and multiscale SABR models
- Geometric interpretation of SABR models
- Calibration problem
- Some numerical results
- Future work
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1.3.2 Notations

- x_t forward rate/forward asset price at time t.
- $E(\cdot)$ the expected value of $\cdot$.
- SABR model:

  (a) v_t stochastic volatility associated to the forward rate/ forward asset price x_t at time t;

  (b) ε volatility of volatility (real parameter).

- Multiscale SABR model:

(a) $v_{i,t}$, $i = 1, 2$, stochastic volatilities associated to the forward rate/ forward asset price x_t at time t;

(b) ε_i, $i = 1, 2$, volatilities of volatilities (real parameters).

- Hull and White model:

(a) v_t stochastic volatility associated to the asset price x_t at time t;

(b) r drift of the asset price, μ drift of the variance $V_t = v_t^2$, ε volatility of volatility (real parameters).

1.3.3 Motivations and Background

A very popular model to price interest rates and foreign exchange derivatives is the SABR model introduced in Hagan et al. (2002), Wilmott Magazine, September 2002, 84-108.

We study:

1. the SABR model and one of its generalizations called SABR model introduced in Fatone et al. (2013) Journal of Mathematical Finance.
2. the Hull and White model introduced in Hull, White(1987), The Journal of Finance, 42, (1987), 281-300.

Note that the log-normal SABR model is a special case of the Hull and White model.

1.3.4 SABR Model

The SABR model describes the dynamics of the forward price of an asset x_t, $t > 0$, and of its stochastic volatility v_t, $t > 0$, (one factor volatility model), through the following stochastic differential equations:

$$dx_t = |x_t|^\beta v_t \, dW_{x,t}, \; t > 0,$$

$$dv_t = \varepsilon \, v_t \, dW_{v,t}, \; t > 0,$$

with initial conditions:

$$x_0 = x^0, v_0 = v^0,$$

where ε, β are real parameters, $\varepsilon > 0$ is the vol of vol, $\beta \in [0, 1]$ is the β-volatility and x^0, v^0 are given random variables. We assume that x^0, v^0 are concentrated in a point with probability one and that $v^0 > 0$. Moreover $W_{x,t}$, $W_{v,t}$,$t > 0$, are stanard Wiener processes such that $W_{x,0} = W_{v,0} = 0$, and $dW_{x,t}$, $dW_{v,t}$, $t > 0$, are their stochastic differentials whose correlation structure is given by:

$$E\left(dW_{x,t} dW_{v,t}\right) = \rho \, dt,$$

where $\rho \in (-1, 1)$ is the correlation coefficient. The models with $\beta = 0$ and $\beta = 1$ are called respectively normal and lognormal SABR models.

1.3.5 Multiscale SABR Model

The multiscale SABR model describes the dynamics of the forward price x_t, $t > 0$, of an asset and of its two stochastic volatilities $v_{1,t}$, $v_{2,t}$, $t > 0$, (two factor volatility model), through the following stochastic differential equations:

$$dx_t = |x_t|^\beta \left(v_{1,t} \, dW_{x,1,t} + v_{2,t} \, dW_{x,2,t}\right), \; t > 0,$$

$$dv_{1,t} = \varepsilon_1 \, v_{1,t} \, dW_{v_1,t}, \; t > 0,$$

$$dv_{2,t} = \varepsilon_2 \, v_{2,t} \, dW_{v_2,t}, \; t > 0,$$

with initial conditions:

$$x_0 = x^0, \quad v_{1,0} = v_1^0, \quad v_{2,0} = v_2^0$$

where the parameters $\varepsilon_i > 0$, $i = 1, 2$ are the vols of vols, $\beta \in [0, 1]$ is the β-volatility and x^0, v_1^0, v_2^0 are given random variables that we assume to be concentrated in a point with probability one. We assume $v_1^0, v_2^0 > 0$. Moreover $dW_{x,v_1,t}$, $dW_{x,v_2,t}$, $dW_{v_1,t}$, $dW_{v_2,t}$, $t > 0$, are the differentials of four standard Wiener processes whose correlation structure is given by:

$$E\left(dW_{x,v_i,t} dW_{v_i,t}\right) = \rho_i dt, i = 1, 2, \; E\left(dW_{x,v_1,t} dW_{x,v_2,t}\right) = 0,$$

$$E\left(dW_{v_1,t} dW_{v_2,t}\right) = 0$$

where $\rho_i \in (-1, 1)$, $i = 1, 2$, are the correlation coefficients. The models with $\beta = 0$ and $\beta = 1$ are called respectively normal and lognormal multiscale SABR models.

1.3.6 Multiscale SABR Model (cont.)

Dropping the $v_{1,t}$ (or $v_{2,t}$) variable, the corresponding equation and initial condition the multiscale SABR model:

$$dx_t = |x_t|^\beta \left(v_{1,t} \, dW_{x,1,t} + v_{2,t} \, dW_{x,2,t}\right), \; t > 0,$$

$$dv_{1,t} = \varepsilon_1 \, v_{1,t} \, dW_{v_1,t}, \; t > 0,$$

$$dv_{2,t} = \varepsilon_2 \, v_{2,t} \, dW_{v_2,t}, \; t > 0,$$

with initial conditions:

$$x_0 = x^0, \quad v_{1,0} = v_1^0, \quad v_{2,0} = v_2^0,$$

reduces to the SABR model.

The stochastic processes $v_{1,t}$, $v_{2,t}$, $t > 0$, are the two factors used to model the volatility of the forward rate/price x_t, $t > 0$.

The use of two factors instead of one factor to model the asset price volatility produces a very careful description of asset price behaviours characterized by two time scales such as short-medium time scales (spikes, abrupt price changes), or medium-long time scale (interest rate term structures). The ratio $\varepsilon_2/\varepsilon_1$ determines the two time scales of the forward rate/price volatility (see, for example, Fatone et al. European Financial Management 2013).

1.3.7 Hull and White Stochastic Volatility Model

The Hull and White model (Hull 1987) describes the dynamics of the asset price x_t, $t > 0$, and of its stochastic volatility v_t, $t > 0$, (one factor volatility model), through the following stochastic differential equations:

$$dx_t = r\, x_t dt + x_t\, v_t\, dW_{x,t},\ t > 0,$$

$$dv_t = \frac{1}{2}\left(\mu - \varepsilon^2\right) v_t\, dt + \varepsilon\, v_t\, dW_{v,t},\ t > 0,$$

with initial conditions:

$$x_0 = x^0, v_0 = v^0,$$

where r, ε, μ are real parameters, r is the drift of the asset, price$\varepsilon > 0$ is thevol of vol, μ is the drift of the variance $V_t = v_t^2$, $t > 0$, and x^0, v^0 are given random variables. We assume that x^0, v^0 are concentrated in a point with probability one and that $v^0 > 0$. Moreover $W_{x,t}$, $W_{v,t}$,$t > 0$, are standard Wiener processes such that $W_{x,0} = W_{v,0} = 0$, and $dW_{x,t}$,

$dW_{v,t}$, $t > 0$, are their stochastic differentials whose correlation structure is given by:

$$E\left(dW_{x,t}dW_{v,t}\right) = \rho\, dt,$$

where $\rho \in (-1, 1)$ is the correlation coefficient. The Hull and White model reduces to the lognormal SABR model (i.e. $\beta = 1$ SABR model) when $\mu = \varepsilon^2$ and $r = 0$.

1.3.8 Some Open Problems about the SABR and the Hull and White Models

Find:

(a) Closed form formulae for the joint probability density function of the state variables of the SABR model when $\beta = 0$, $\beta = 1$ and of the Hull and White model in presence of nonzero correlation, that is when $\rho \neq 0$;

(b) Closed form formulae for the joint probability density function of the state variables of the SABR model when $\beta \in (0, 1)$ in the case of zero correlation, that is when $\rho = 0$;

(c) Closed form formulae to price European call and put options for the models specified in $a)$ and $b)$;

(d) Closed form formulae analogous to those of (a), (b) and (c) for the multiscale versions of the SABR and of the Hull and White models.

1.3.9 Last Year Results

Last year results were limited to the normal SABR and multiscale SABR models.

1. Closed form formulae for the joint transition probability density functions of the normal SABR and multiscale SABR models. The knowledge of these joint transition probability density functions in closed form allowed us to derive formulae for the corresponding option prices and for the moments of the asset price process.

2. The formula for the joint transition probability density function of the normal SABR model has been expressed through a "kernel" related to the heat kernel of the Kontorovich-Lebedev transform.

3. Under suitable assumptions on the correlation structure (absence of correlation between the volatilities) the joint probability density function of the normal multiscale SABR model has been expressed as a kind of "convolution" of two copies of the kernel of the normal SABR model.

1.3.10 This Year Results on SABR and Hull and White Models and on Their Multiscale Versions

We have obtained:

1. New closed form formulae for the joint transition probability density function of:

   - log-normal SABR model (i.e. $\beta = 1$) in presence of nonzero correlation, $\rho \in (-1, 1)$ (Fourier transform, heat kernel of the Kontorovich-Lebedev transform);
   - Hull and White model in presence of nonzero correlation, $\rho \in (-1, 1)$ (Fourier transform, heat kernel of the index Whittaker transform);

- SABR model with $\beta \in (0, 1)$, in the case of zero correlation, $\rho = 0$ (Hankel transform, heat kernel of the Kontorovich-Lebedev transform) .

2. New expansion in powers of ρ of the joint transition probability density function of the SABR model with $\beta \in (0, 1)$, $\rho \in (-1, 1)$.

3. New formulae for the joint transition probability density functions of the multiscale versions of the models mentioned in 1), 2), 3) under suitable assumptions on their correlation structure using a kind of convolution of the kernels mentioned above.

4. New formulae for the geodesic curves of the Riemannian manifolds associated to the SABR models.

1.3.11 This Year Results on SABR and Hull and White Models and on Their Multiscale Versions (cont.)

As a consequence of the results listed in the previous slide, we obtain new closed form formulae for European call and put option prices in the following models:

1. Log-normal SABR model (i.e. $\beta = 1$) in presence of nonzero correlation, $\rho \in (-1, 1)$;

2. Hull and White model in presence of nonzero correlation, $\rho \in (-1, 1)$;

3. SABR model with $\beta \in (0, 1)$, in the case of zero correlation, $\rho = 0$;

4. Multiscale versions of the normal and lognormal SABR and of the Hull and White models.

Note that for the SABR model when $\beta \in (0,1)$, $\rho \in (-1,1)$ we have deduced expansions in powers of ρ with base point $\rho = 0$ for the European call and put option prices.

The SABR model with $\beta = 1/2$ can be seen as a stochastic volatility version of the CIR model. This model deserves special attention and will be investigated elsewhere.

1.3.12 The Heat Kernel: An Example

Let us explain the meaning of "heat kernel" of an integral transform in the simplest case. That is in the case when the Wiener process and the heat equation are studied using the Fourier transform. Let us consider the stochastic differential equation:

$$dx_t = \sigma \, dW_t, \ \ t > 0,$$

where dW_t is the differential of the standard Wiener process W_t, $t > 0$, and $\sigma > 0$ is a constant. The solution x_t, $t > 0$, of this differential equation is a Wiener process and its transition probability density function $p(x_f, t_f, x, t)$ is the probability of having at time $t_f > 0$ $x_{t_f} = x_f$ given that at time $t \geq 0$ we have $x_t = x$ when $t < t_f$. The function p is the solution of the following backward Kolmogorov equation:

$$-\frac{\partial p}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}, \ x \in (-\infty, +\infty), \ \ t < t_f,$$

with final condition:

$$p(x_f, t_f, x, t_f) = \delta(x - x_f), \quad x, x_f \in (-\infty, +\infty).$$

Note that the previous backward Kolmogorov equation is a back-ward heat equation. To determine p we must solve this final value problem.

1.3.13 Solution Procedure

Step 1 Change the independent variables t and x using the invariance properties of the function p implied by the backward Kolmokorov equation ($s = t_f - t, t < t_f, \xi = x - x_f, x, x_f \in (-\infty, +\infty)$).

Step 2 Assume that $p^*(s, \xi) = p(x_f, t_f, x, t)$, $s = t - t_f$, $\xi = x - x_f$ has the following form:

$$p^*(t_f - t, x - x_f) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{-\imath k (x - x_f)} g(t_f - t, k).$$

The function g is the Fourier transform of p^* with respect to the variable $\xi = x - x_f$. We have:

$$g(s, k) = \int_{-\infty}^{+\infty} d\xi\, e^{\imath k \xi}\, p^*(s, \xi),$$

Step 3 Determine g as the solution of the backward Kolmogorov equation written in the new independent variables (s, k). Note that in the case studied here the Fourier transform diagonalizes the differential operator in the variable x of the backward Kolmogorov equation.

The function g is the "heat kernel".

1.3.14 The Heat Kernel - Step 1

Step 1: The function p is invariant by time translation. Let $p^*(s, x - x_f) = p(x_f, t_f, x, t)$, $s = t_f - t > 0$, $x, x_f \in (-\infty, +\infty)$.

The backward Kolmogorov equation can be rewritten as follows:

$$\underbrace{\frac{\partial p^*}{\partial s} = \frac{\sigma^2}{2}\frac{\partial^2 p^*}{\partial x^2}}_{heat\, equation},\ x \in (-\infty, +\infty), s > 0,$$

with initial condition (the Dirac's delta function δ):

$$p^*(0, x_f - x) = \delta(x - x_f), x, x_f \in (-\infty, +\infty).$$

Note that p^* is the fundamental solution of the heat equation since it satisfies the heat equation and the initial condition given by the (impulse) Dirac's delta function.

1.3.15 The Heat Kernel - Step 2-3

Step 2: Take p^* in the form:

$$p^*(s, x - x_f) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{-\imath\, k\,(x - x_f)} g(s, k).$$

The function g is the Fourier transform of p^* with respect to the variable $\xi = x - x_f$.

Step 3: In order to satisfy the initial condition we have:

$$\delta(x - x_f) = p^*(0, x_f, x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{-\imath\, k\,(x - x_f)} g(0, k),$$

this implies $g(0, k) = 1$, $k \in (-\infty, +\infty)$. In fact we have:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{-\imath\, k\, x}\int_{-\infty}^{+\infty} d\eta\, e^{+\imath\,\eta\, k}\, f(\eta),\ \ \forall\, f$$

that is:

$$\delta(x-\eta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, e^{-\imath\, k\, x}\; e^{\imath\, k\, \eta}$$

These last two formulae are called resolution of the identity of the Fourier transform.

1.3.16 The Heat Kernel - Step 3

The function g satisfies the following initial value problem that translates the final value problem for the backward Kolmogorov equation written above:

$$\underbrace{\frac{\partial g}{\partial s} = -k^2 \frac{\sigma^2}{2} g}_{the\, spatial\, part\, has\, been\, diagonalized} \qquad , \; k \in (-\infty, +\infty), s > 0,$$

with initial condition:

$$g(0,k) = 1, k \in (-\infty, +\infty).$$

The equation satisfied by g is an ordinary differential equation depending on the parameter k. The solution of the initial value problem satisfied by g is:

$$g(s,k) = e^{-k^2 s/(2\sigma^2)}, \quad k \in (-\infty, +\infty), s > 0,$$

and we have:

$$p^*(s, x - x_f) = \frac{e^{-\frac{\left(x_f - x\right)^2}{2\sigma^2\, s}}}{\sigma\,\sqrt{2\pi\, s}} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\, \underbrace{e^{-\sigma^2 k^2 s/2}}_{heatkernel} e^{-\imath\, k\, x}\; e^{\imath\, k\, x_f},$$

$$x \in (-\infty, +\infty), s > 0.$$

1.3.17 The Heat Kernel

The procedure used in the previous slides to derive the transition probability density function of the Wiener process is used to obtain the transition probability density functions of the SABR and Hull and White models.

Note that the presence of the stochastic volatility (or volatilities) in these models implies that after taking the Fourier transform (or the Hankel transform) with respect to the asset price variable the equation satisfied by g is still a partial differential equation of parabolic type in the time and the stochastic volatility variable (or variables).

In order to find an ”explicit” solution of this last equation we use another integral transform that ”diagonalizes” the elliptic part of the parabolic equation satisfied by g. That is the Kontorovich-Lebedev transform or the index Whittaker transform.

1.3.18 The Transforms Corresponding to the Kernels Used in the Study of the SABR and the Hull and White Models

The Kontorovich-Lebedev transform ($\beta = 0$, $\beta = 1$ SABR model):

$$K_f(x) = \int_0^{+\infty} d\omega\, f(\omega) K_{\imath\omega}(x)$$

The index Whittaker transform (Hull and White model):

$$W_f^{\mu}(x) = \int_0^{+\infty} d\omega\omega\, \sinh(2\pi\omega) f(\omega) W_{\mu,\imath\omega}(x)$$

The Hankel transform+Kontorovich-Lebedev transform

($\beta \in (0,1)$, $\rho = 0$ SABR model):

$$\text{The Hankel transform} : B_f^{\mu}(x) = \int_0^{+\infty} d\omega \omega\, f(\omega) J_{\mu}(\omega\, x)$$

where $\imath$ is the imaginary unit, K_ν, $W_{\mu,\nu}$, J_μ are respectively the modified Bessel function of second kind of index ν, the Whittaker function of indices μ, ν and the Bessel function of first kind of index μ.

1.3.19 Resolutions of the Identity Associated to the Previous Integral Transforms

Kontorovich-Lebedev resolution of the identity (Yakubovic 2011):

$$f(x) = \frac{2}{\pi^2} \int_0^{+\infty} d\omega\, \omega \sinh(\pi\omega) K_{\imath\omega}(x) \int_0^{+\infty} \frac{d\eta}{\eta} K_{\imath\omega}(\eta) f(\eta), \forall f,$$

Index Whittaker resolution of the identity (Szmytkowski, Bielski 2010):

$$f(x) = \frac{1}{\pi^2} \int_0^{+\infty} d\omega\, \omega \sinh(2\pi\omega) \Gamma\left(\frac{1}{2} - \nu + \imath\omega\right) \Gamma\left(\frac{1}{2} - \nu - \imath\omega\right) \cdot$$

$$W_{\nu,\imath\omega}(x) \int_0^{+\infty} \frac{d\eta}{\eta^2} W_{\nu,\imath\omega}(\eta) f(\eta), \forall f,$$

The Hankel resolution of the identity:

$$f(x) = \int_0^{+\infty} d\omega\, \omega\, J_\nu(\omega\, x) \int_0^{+\infty} d\eta\, \eta J_\nu(\omega\, \eta) f(\eta),\ \forall f,$$

where Γ is the Gamma function.

1.3.20 This Year Results on SABR and Hull and White Models (cont.)

The different nature of the kernels behind the SABR models with $\beta = 0$ or $\beta = 1$ (i.e. Fourier transform and Kontorovich-Lebedev transform) and the SABR models with $\beta \in (0, 1)$ (i.e. Hankel transform and Kontorovich-Lebedev transform) is due to the fact that the second order differential operator of the backward Kolmogorov equation associated to the SABR model with $\beta = 0$ or $\beta = 1$ is a differential operator in the conjugate variable of the Fourier transform of the asset price, while this is not true when $\beta \in (0, 1)$.

The heat kernels considered previously for the SABR models are related to the fundamental solutions of the "heat equation" on suitable Riemannian manifolds. The elliptic second order operators appearing in these heat equations are the "Laplace-Beltrami operators" of the Riemannian manifolds. The study of the geodesic curves of these manifolds and of their length can be used to build approximations of the solutions of the corresponding "heat equations" or of equations related to them as the backward Kolmogorov equations of the SABR models.

1.3.21 This Year Results on SABR and Hull and White Models

P. Hagan, A. Lesniewski and D. Woodward (2005) studied the differential geometry of the SABR model and derived an expression of the geodesic distance of the Riemannian geometry associated to the SABR model. We carry one step further their analysis and we obtain the analytical expression of the corresponding geodesic curves. We study the behaviour of these geodesic curves as a function of several parameters.

We show that under some assumptions on their correlation structure the joint probability density functions of the multiscale SABR and Hull and White models are a "convolution" of two copies of the kernel of the corresponding non multiscale models.

1.3.22 SABR Backward Kolmogorov Equation

$$dx_t = |x_t|^\beta \, v_t \, dW_{x,t}, \; t > 0,$$

$$dv_t = \varepsilon \, v_t \, dW_{v,t}, \; t > 0.$$

The joint transition probability density function $p_\beta(x_f, v_f, t_f, x, v, t)$ of the state variables of the normal SABR model is the probability of having at time $t_f > 0$ $(x_{t_f}, v_{t_f}) = (x_f, v_f)$ given that at time $t \geq 0$ we have $(x_t, v_t) = (x, v)$ when $t < t_f$. The function p_β is invariant for time translation so that it can be rewritten as a function of the difference $s = t_f - t$ and it is the solution of the following problem (backward Kolmogorov equation):

$$\frac{\partial p_\beta}{\partial s} = \frac{1}{2} x^{2\beta} v^2 \frac{\partial^2 p_\beta}{\partial x^2} + \frac{\varepsilon^2}{2} v^2 \frac{\partial^2 p_\beta}{\partial v^2} + \rho \, v^2 x^\beta \frac{\partial^2 p_\beta}{\partial \, x \partial \, v},$$

$$s = t_f - t > 0, \; x \in (-\infty, +\infty), \, v \in (0, +\infty),$$

$$p_\beta(s = 0, x_f, v_f, x, v) = \delta(x - x_f) \, \delta(v - v_f) \, .$$

Note that when $\beta = 0$ and $\beta = 1$ we have integer powers of the variable x in the coefficients of the previous equation and this makes the normal ($\beta = 0$) and the lognormal ($\beta = 1$) SABR models easier to treat analytically than the models with $\beta \in (0, 1)$.

1.3.23 Solution Procedure

Step 1 Change the independent variables t and x using the invariance properties of the backward Kolmokorov equation, that is introduce the variables $s = t_f - t$, and $x = f_\beta(\xi)$, $\xi \in (-\infty, +\infty)$, when $\beta = 0, 1$, or $\xi \in (0, +\infty)$, when $\beta \in (0, 1)$ where f_β is a suitable function depending on the parameter $\beta \in [0, 1]$;

Step 2 Determine $p^*_\beta(\xi_f, v_f, t_f, \xi, v, t)$, $t < t_f$, such that $p_\beta(x_f, v_f, t_f, x, v, t)dx_f\, dv_f = p^*_\beta(\xi_f, v_f, t_f, \xi, v, t) f'_\beta(\xi_f) d\xi_f\, dv$, where f'_β is the derivative of f_β and assume that p^* has the following form:

$$p^*_\beta(\xi_f,v_f,t_f,\xi,v,t) = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-\imath k(\xi-\xi_f)} g_\beta(t_f - t, k, v_f, v)}_{(Fourier\, Transform)}, \beta = 0,1,$$

$$p^*_\beta(\xi_f,v_f,t_f,\xi,v,t) = \underbrace{\int_0^{\infty} dk\, k J_{\frac{1}{2(1-\beta)}}(\xi\, k) J_{\frac{1}{2(1-\beta)}}(\xi_f\, k) g_\beta(t_f - t, k, v_f, v)}_{(Hankel\, Transform)},$$
$$\beta \in (0, 1),$$

Step 3 Determine g_β as the solution of the backward Kolmogorov equation rewritten in the new independent variables (s, k, v).

The function g_β is the "heat kernel" of the SABR model.

1.3.24 The Heat Kernel of SABR Model $\beta = 0$, $\beta = 1$

Transforms:

$$\beta = 0 \Longrightarrow x = f_0(\xi) = \xi, \xi \in (-\infty, +\infty),$$

$$\beta = 1 \Longrightarrow x = f_1(\xi) = e^{\xi}, \xi \in (-\infty, +\infty),$$

Partial differential equation for $g_\beta(s, k, v_f, v)$, $k \in (-\infty, \infty)$, $s, v, v_f \in (0, +\infty)$, $\beta = 0, 1$:

$$\frac{\partial g_\beta}{\partial s} = \frac{\varepsilon^2}{2} v^2 \frac{\partial^2 g_\beta}{\partial v^2} - \imath\, k\, \rho\, v^2 \frac{\partial\, g_\beta}{\partial\, v} + \left(-\frac{1}{2} k^2 v^2 + \beta \left(\frac{\imath\, k}{2} \right) \right) g_\beta,$$

$$s = t_f - t > 0,\ k \in (-\infty, \infty), v \in (0, +\infty),\ \beta = 0, 1.$$

with initial condition:

$$g_\beta(0, k, v_f, v) = \delta(v - v_f),\ k \in (-\infty, \infty), v \in (0, +\infty),\ \beta = 0, 1,$$

where δ is the Dirac's delta function.

1.3.25 The Heat Kernel of SABR Model $\beta = 0$, $\beta = 1$ (cont.)

Using the Kontorovich-Lebedev transform and the corresponding resolution of the identity it can be shown that the function g_β, $\beta = 0, 1$, has the following form:

$$g_\beta(s,k,v_f,v,\varepsilon,\rho)= \frac{2}{\pi^2}\, e^{-\frac{s}{8}\varepsilon^2} \left(\frac{\sqrt{v}}{\sqrt{v_f} v_f}\right) e^{\imath\, k\, \rho (v-v_f)/\varepsilon} \,.$$

$$\int_0^{+\infty} d\omega\, \underbrace{e^{-s\varepsilon^2\omega^2/2}\omega \sinh(\pi\omega) K_{\imath\omega}\left(\nu_\beta(k) v_f\right) K_{\imath\omega}\left(\nu_\beta(k) v\right)}_{heat\ kernel\ of\ Kontorovich-Lebedev\, transform},$$

$$s>0, k\in(-\infty,+\infty), v_f, v\in(0,+\infty), \varepsilon>0, \rho\in(-1,1), \beta=0,1,$$

where

$$\nu_\beta(k) = \left(\frac{k^2}{\varepsilon^2} - \imath\,\beta\,\frac{k}{\varepsilon^2}\right)^{1/2},\ k\in(-\infty,+\infty),\ v_f, v\in(0,+\infty),$$

$$\varepsilon>0,\ \rho\in(-1,1), \beta=0,1.$$

1.3.26 The Joint Transition Probability Density Function of SABR Model $\beta = 0$, $\beta = 1$ (cont.)

$$p^*_\beta(\xi_f, v_f, t_f, \xi, v, t) =$$

$$\frac{1}{\sqrt{\pi}\pi} \; \frac{1}{\sqrt{2(t_f - t)\varepsilon^2}} \, e^{-\frac{(t_f-t)}{8}\varepsilon^2} \, e^{\frac{\pi^2}{2(t_f-t)\varepsilon^2}} \; \frac{(1-\rho^2)}{\varepsilon^2} \; \left(\frac{v\sqrt{v}}{\sqrt{v_f}}\right) \cdot$$

$$e^{\beta(-(\xi-\xi_f)+\rho(v-v_f)/\varepsilon)/(2(1-\rho^2))} \cdot$$

$$\int_0^{+\infty} du \, \sinh(u) \sin\left(\frac{u\pi}{(t_f - t)\varepsilon^2}\right) e^{-\frac{u^2}{2(t_f-t)\varepsilon^2}} \cdot$$

$$e^{-\frac{\beta}{2(1-\rho^2)}\left[(-(\xi-\xi_f)+\rho(v-v_f)/\varepsilon)^2 + \frac{(1-\rho^2)}{\varepsilon^2}(v^2+v_f^2+2v_f v\cosh(u))\right]^{\frac{1}{2}}} \cdot$$

$$\left\{ \frac{1}{\left[\left(-(\xi-\xi_f) + \frac{\rho}{\varepsilon}(v - v_f)\right)^2 + \frac{(1-\rho^2)}{\varepsilon^2}(v_f^2 + v^2 + 2v_f v\cosh(u))\right]^{3/2}} \right.$$

$$\left. + \frac{\beta/(2(1-\rho^2))}{\left[\left(-(\xi-\xi_f) + \frac{\rho}{\varepsilon}(v - v_f)\right)^2 + \frac{(1-\rho^2)}{\varepsilon^2}(v_f^2 + v^2 + 2v_f v\cosh(u))\right]} \right\},$$

$t < t_f, \xi, \xi_f \in (-\infty, +\infty), v, v_f \in (0, +\infty), \varepsilon > 0, \rho \in (-1, 1), \beta = 0{,}1.$

1.3.27 Option Pricing in the SABR Model $\beta = 0$, $\beta = 1$

Let us define the function $\mathcal{S}_\beta$:

$$\mathcal{S}_\beta(s,k,v,\varepsilon,\rho) = \int_0^{+\infty} dv_f g_\beta(s,k,v_f,v)$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} e^{-\frac{s}{8}\varepsilon^2} e^{\imath\, k\frac{\rho\, v}{\varepsilon}} \frac{e^{\frac{\pi^2}{2s\varepsilon^2}}}{\sqrt{2s\varepsilon^2}}$$

$$\int_0^{+\infty} du \sinh(u)\sin\left(\frac{\pi u}{s\varepsilon^2}\right) e^{-u^2/(2s\varepsilon^2)} \, .$$

$$\int_0^{+\infty} \frac{dy}{\sqrt{y}} e^{-y\cosh(u)}\, e^{-\tilde{a}_\beta(y)}$$

$$s \in \mathbb{R}^+,\ k \in \mathbb{R},\ \ v \in \mathbb{R}^+, \varepsilon > 0, \rho \in (-1,1),$$

where $\tilde{a}^2_\beta$ is given by:

$$\tilde{a}^2_\beta(y) = \left(y^2 + v^2 k^2 \frac{(1-\rho^2)}{\varepsilon^2}\right) + \imath\, k \frac{v}{\varepsilon}\left(2y\rho - \beta\frac{v}{\varepsilon}\right),$$

$$y \in \mathbb{R}^+, \beta = 0, 1.$$

1.3.28 European Call Option Price in the SABR and Multiscale SABR Models with $\beta = 0, 1$

In the normal ($\beta = 0$) and lognormal ($\beta = 1$) SABR and multiscale SABR models with risk neutral parameters let $C_\beta(x, K, T)$ and $C_{M,\beta}(x, K, T)$ be respectively the prices at time $t = 0$ in the SABR and in the multiscale SABR models of an European call option on the asset whose forward price is $x = f_\beta(\xi)$, $\beta = 0, 1$ (see Step 1), having expiry date $T > 0$ and strike price K, we have:

$$C_\beta(x,K,T) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\xi'(f_\beta(\xi') - K)_+$$

$$\int_{-\infty}^{+\infty} d\eta e^{-\imath\eta(\xi-\xi')}\mathcal{S}_\beta(T,\eta,v^0,\varepsilon,\rho),$$

$$\xi \in (-\infty,+\infty),\ v^0, T > 0,$$

$$C_{M,\beta}(x,K,T) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\xi'(f_\beta(\xi') - K)_+$$

$$\int_{-\infty}^{+\infty} d\eta e^{-\imath\eta(\xi-\xi')} \underbrace{\mathcal{S}_\beta(T,\eta,v_1^0,\varepsilon_1,\rho_1)\mathcal{S}_\beta(T,\eta,v_2^0,\varepsilon_2,\rho_2)}_{kind\,of\,convolution},$$

$$\xi \in (-\infty,+\infty), v_1^0, v_2^0, T > 0,$$

where $(\,\cdot\,)_+$ denotes the maximum between zero and $\cdot$ and $\mathcal{S}_\beta(s,\eta,v,\varepsilon,\rho)$ is the function shown in the previous slide. These option pricing formulae are used in the numerical experiments discussed at the end.

1.3.29 The Joint Transition Probability Density Function of Hull and White Model

$$p_{HW}(x_f,v_f,t_f,x,v,t) = \frac{1}{2\pi}\frac{1}{x_f}\int_{-\infty}^{+\infty} dx\, e^{-\imath\, k(\ln(x/x_f))} g_{HW}(t-t',k,v_f,v),$$

$$(x,v),\ (x_f,v_f) \in (0,+\infty)\times(0,+\infty),\ t,t_f \geq 0, t < t_f,$$

the function g_{HW} is given by:

$$g_{HW}(s,k,v_f,v) = \frac{1}{\pi^2} e^{-\frac{\varepsilon^2 s}{8}} e^{-(\tilde{\mu}^2-2\tilde{\mu})\frac{\varepsilon^2 s}{8}} e^{-\imath\frac{k}{\varepsilon}\rho(v-v_f)} e^{-\frac{\tilde{\mu}}{2}\ln(v/v_f)} \cdot$$
$$\frac{1}{2\nu(k)^{1/2}} \frac{1}{v_f^2} \int_0^{+\infty} d\omega \cdot$$
$$\underbrace{e^{-\frac{s\varepsilon^2\omega^2}{2}} \omega \sinh(2\pi\omega) \Gamma\left(\frac{1}{2} - a(k) + \imath\omega\right) \Gamma\left(\frac{1}{2} - a(k) - \imath\omega\right) W_{a(k)\imath\omega}\left(2v\nu(k)^{1/2}\right) W_{a(k)\imath\omega}\left(2v_f\nu(k)^{1/2}\right)}_{heat\,kernel\,of\,index\,Whittaker\,transform}$$

where

$$\nu(k) = \frac{k^2}{\varepsilon^2}(1-\rho^2) - \frac{\imath k}{\varepsilon^2},\ a(k) = \imath\frac{\tilde{\mu}}{2}\frac{\rho}{\varepsilon}\frac{k}{\nu(k)^{1/2}},\ k \in (-\infty, +\infty).$$

(for further details Fatone et al 2013, International Journal of Modern nonlinear Theory and Application.)

1.3.30 The Heat Kernel of the SABR Model $\beta \in (0,1)$, $\rho = 0$

We rewrite the joint probability density function function $p_\beta(x_f, v_f, t_f, x, v, t)$, $x, x_f, v, v_f \in (0, +\infty)$, $t < t_f$, $\beta \in (0,1)$, $\rho = 0$, using the change of variable $x = f_\beta(\xi) = (1-\beta)^{1/(1-\beta)}\xi^{1/(1-\beta)}$. We deduce the following expression of the joint probability density function in the new variable $p^*_\beta(s, \xi_f, v_v, \xi, v) = p_\beta(f_\beta(\xi_f), v_f, t_f, f_\beta(\xi), v, t)$, $s = t_f - t$:

$$p^*_\beta(s,\xi_f,v_f,\xi,v)=(1-\beta)^{-1/(1-\beta)}\xi_f^{-1/(1-\beta)}\xi_f^{1-\nu}\,\xi^\nu\int_0^{+\infty}d\lambda\,\lambda\cdot$$

$$\Bigg\{\underbrace{J_{\frac{1}{2(1-\beta)}}(\xi_f\,\lambda)\,J_{\frac{1}{2(1-\beta)}}(\xi\,\lambda)}_{Hankel\,transform}\frac{2}{\pi^2}\,e^{-\frac{s}{8}\varepsilon^2}\left(\frac{\sqrt{v}}{\sqrt{v_f v_f}}\right)\int_0^{+\infty}d\omega\cdot$$

$$\left.\underbrace{e^{-s\varepsilon^2\omega^2/2}\omega\sinh(\pi\omega)K_{\imath\omega}\,(\lambda\,v_f)\,K_{\imath\omega}\,(\lambda\,v)}_{heat\;kernel\;of\;Kontorovich-Lebedev\,transform}\right\},$$

$$s>0,\;k\in(-\infty,+\infty),\;\;\xi_f,\xi,\;v_f,v\in(0,+\infty)\,.$$

1.3.31 Riemannian Metric Associated to the SABR Models $\beta \in [0,1)$

Let us consider H_0={ $(x,v)\in\mathbb{R}\times\mathbb{R}^+$ } and H_β={ $(x,v)\in\mathbb{R}^+\times\mathbb{R}^+$ }, $\beta\in(0,1)$. The set H_β is equipped with a Riemannian metric depending on $\beta\in[0,1)$ given by:

$$G=((g_{i,j}))=\frac{1}{v^2\varepsilon^2x^{2\beta}\sqrt{1-\rho^2}}\begin{pmatrix}\varepsilon^2 & -\rho\,|x|^\beta\,\varepsilon\\ -\rho\,|x|^\beta\,\varepsilon & x^{2\beta}\end{pmatrix},$$

$$G^{-1}=((g^{i,j}))=\frac{v^2}{\sqrt{1-\rho^2}}\begin{pmatrix}x^{2\beta} & \rho\,|x|^\beta\,\varepsilon\\ \rho\,|x|^\beta\,\varepsilon & \varepsilon^2\end{pmatrix}.$$

The Laplace Beltrami operator Δ_G on the Riemannian manifold H_β with the metric tensor g associated to the metric G is given by:

$$\Delta_G\,p_\beta\;=\;\frac{1}{\sqrt{g}}\sum_{\mu=1}^{2}\frac{\partial}{\partial\,x^\mu}\left(\sqrt{g}\sum_{\nu=1}^{2}g^{\mu,\nu}\frac{\partial p_\beta}{\partial\,x^\nu}\right),$$

where we have used the notation $x^1=x$, $x^2=v$ and g is the determinant of G.

The backward Kolmogorov equation of the SABR model can be written as follows:

$$\frac{\partial p_\beta}{\partial s} = \frac{1}{2}\Delta_G p_\beta - \frac{\beta}{2}\varepsilon^2 x^{2\beta-1} v^2 \frac{\partial\, p_\beta}{\partial x}$$

1.3.32 Geodesics Equations Associated to the SABR Models $\beta \in [0,1)$

Let τ be the arc length of a curve and let $x(\tau)$, $v(\tau)$, $\tau > 0$ denote a geodesic curve of the manifold H_β, $\beta \in [0,1]$. When $\beta \in [0,1)$ a standard calculation gives the following system of ordinary differential equations for the geodesic curves:

Equation for x=x(τ)

$$\frac{d^2x}{d\tau^2}+\Gamma^1_{1,1}\left(\frac{dx}{d\tau}\right)^2+2\Gamma^1_{1,2}\left(\frac{dx}{d\tau}\right)\left(\frac{dv}{d\tau}\right)+\Gamma^1_{2,2}\left(\frac{dv}{d\tau}\right)^2=0\Longrightarrow$$

$$(1-\rho^2)\frac{d^2x}{d\tau^2} + \left(-\frac{\beta}{x} + \frac{\beta\,\rho^2}{x} + \frac{\varepsilon\,\rho}{|x|^\beta\, v}\right)\left(\frac{dx}{d\tau}\right)^2 -$$

$$2\left(\frac{1}{v}\right)\left(\frac{dx}{d\tau}\right)\left(\frac{dv}{d\tau}\right) + \frac{\rho\,|x|^\beta}{\varepsilon\, v}\left(\frac{dv}{d\tau}\right)^2 = 0$$

<u>Equation for v=v(τ)</u>

$$\frac{d^2v}{d\tau^2}+\Gamma^2_{1,1}\left(\frac{dx}{d\tau}\right)^2+2\Gamma^2_{1,2}\left(\frac{dx}{d\tau}\right)\left(\frac{dv}{d\tau}\right)+\Gamma^2_{2,2}\left(\frac{dv}{d\tau}\right)^2=0\Longrightarrow$$

$$(1-\rho^2)\frac{d^2v}{d\tau^2}+\left(\frac{\varepsilon^2}{v\,x^{2\beta}}\right)\left(\frac{dx}{d\tau}\right)^2-$$

$$2\left(\frac{\varepsilon\,\rho}{v\,|x|^{\beta}}\right)\left(\frac{dx}{d\tau}\right)\left(\frac{dv}{d\tau}\right)+\left(\frac{(2\rho^2-1)}{v}\right)\left(\frac{dv}{d\tau}\right)^2=0.$$

where $\Gamma^k_{i,j}$, $i,j,k=1,2$, are the Christoffel symbols.

1.3.33 Geodesics Associated to the SABR Models $\beta\in[0,1)$

The previous system of ordinary differential equations that characterizes the geodesic curves has been solved explicitly. The geodesic curve that joins the points $P_0=(x_0,v_0)$ and $P_1=(x_1,v_1)$ has the following expression:

$$\left(\varepsilon\,\frac{|x|^{1-\beta}}{1-\beta}-\rho\,v-a\right)^2+(1-\rho^2)v^2=b^2,\ \ \beta\in(0,1).$$

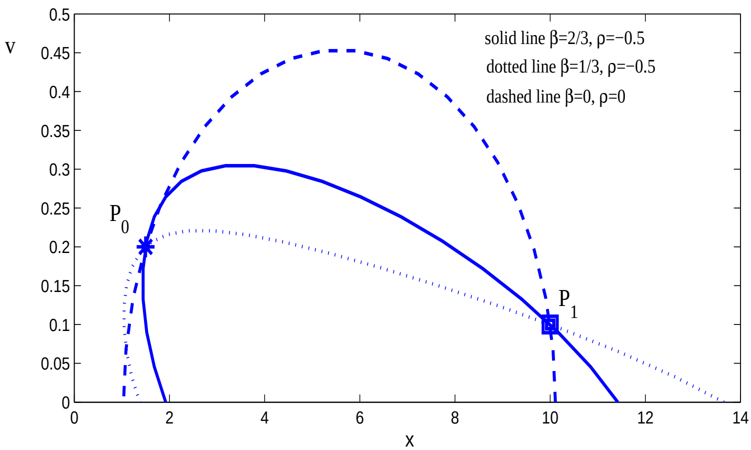
where a and b are given by:

$$a=\frac{1}{2}(\eta_1+\eta_0)+\frac{1}{2}(1-\rho^2)\frac{(v_1^2-v_0^2)}{(\eta_1-\eta_0)},\eta_i=\varepsilon\frac{|x_i|^{1-\beta}}{1-\beta}-\rho\,v_i,i=0,1,$$

$$b^2=\frac{1}{4}(\eta_1-\eta_0)^2+\frac{1}{4}(1-\rho^2)\frac{(v_1^2-v_0^2)^2}{(\eta_1-\eta_0)^2}+\frac{1}{2}(1-\rho^2)(v_1^2+v_0^2)$$

Note that the geodesics are semi-ellipses in the plane (η, v) where $\eta = \varepsilon\, \frac{|x|^{1-\beta}}{1-\beta} - \rho\, v$.

1.3.34 Geodesic Curves Associated to the SABR Models $\beta \in [0, 1)$



Geodesics passing through the points P_0 and P_1

1.3.35 Geodesic Curves Associated to the SABR Models $\beta \in [0, 1)$

The knowledge of explicit formulae for the geodesic curves of the Riemannian manifolds associated to the SABR models can be used to approximate the joint transition probability density functions of these models. There are two ways of doing this:

1) in the small time limit use Varadhan's theorem and the analogous in this context of the Wentzel-Kramers-Brilllouin (WKB) semiclassical approximation in quantum mechanics;

2) develop numerical methods to evaluate the path integral (Wiener integral) formula that gives the transition probability density function.

These numerical methods are kind of importance sampling Monte Carlo methods (that exploit geodesic formulae) to evaluate integrals in infinitely many dimensions.

These questions deserve further attention and will be investigated elsewhere.

1.3.36 Calibration Problem

Many different calibration problems can be considered. Let us study one of them.

Let m be a positive integer and let $C(x_0, K_j, T)$, $P(x_0, K_j, T)$ be the observed prices of the European call and put options with strike prices K_j, $j = 1, 2, \ldots, m$, maturity time T and forward asset price x_0 at time $t = t_0 = 0$ and let $C_Q(x_0, K_j, T)$, $P_Q(x_0, K_j, T)$ be the corresponding prices of the European call and put options computed using respectively the lognormal SABR ($Q = L$) and multiscale SABR ($Q = ML$) models. Let $\mathcal{M}^Q$, $Q = L, ML$, be given by:

$$\mathcal{M}^{ML} = \{\underline{\Theta} = (\varepsilon_1, \rho_{0,1}, \tilde{v}_{0,1}, \varepsilon_2, \rho_{0,2}, \tilde{v}_{0,2}, r) \in R^7 \mid \varepsilon_i > 0,$$

$$-1 < \rho_{0,i} < 1, \tilde{v}_{i,0} \geq 0,\ i = 1, 2,\ \varepsilon_1 \leq \varepsilon_2,\ r \geq 0\}.$$

$$\mathcal{M}^{L} = \{\underline{\Theta} = (\varepsilon, \rho, \tilde{v}_0, r) \in R^4 \mid \varepsilon > 0,\ -1 < \rho < 1,$$

$$\tilde{v}_0 \geq 0,\ r \geq 0\}.$$

Calibration Problem: find an estimate of the vector $\underline{\Theta}$ from the observations, that is, from the knowledge of the observed option prices $C(x_0, K_j, T)$, $P(x_0, K_j, T)$, $j = 1, 2, \ldots, m$.

1.3.37 Formulation of the Calibration Problem

The calibration problem considered is formulated as follows:

$$\min_{\underline{\Theta} \in \mathcal{M}^Q} L_Q(\underline{\Theta}),$$

where the objective function $L_Q(\underline{\Theta})$ is given by:

$$L_Q(\underline{\Theta}) = \sum_{i=1}^{m} \left[C_Q(x_0, K_i, T) - C(x_0, K_i, T)\right]^2 + \sum_{i=1}^{m} \left[P_Q(x_0, K_i, T) - P(x_0, K_i, T)\right]^2,$$
$$\underline{\Theta} \in \mathcal{M}^Q,\ Q = L, ML,$$

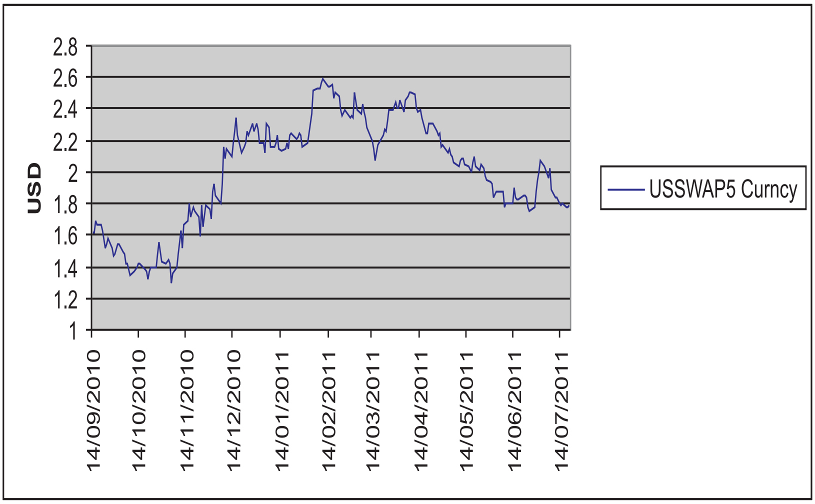
We compute the relative errors:

$$\epsilon_Q(K) = |C_Q(x_0, K, T) - C(x_0, K, T)|/|C(x_0, K, T)|,$$

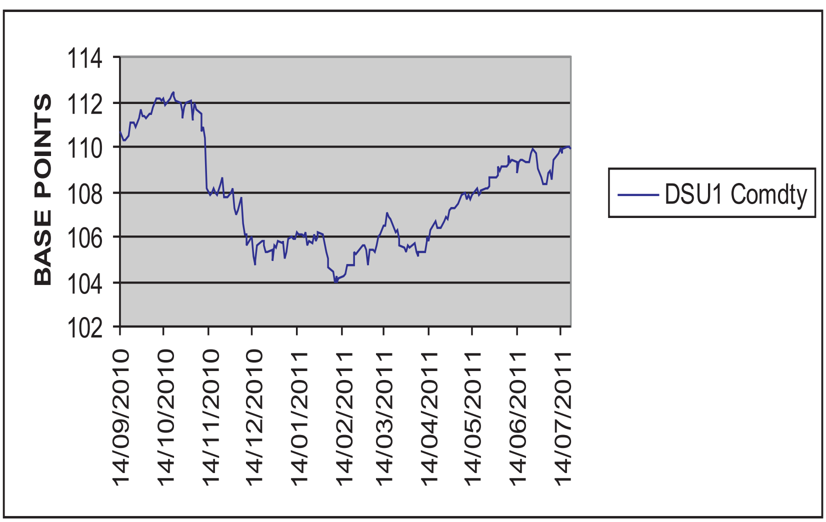
$$\psi_Q(K) = |P_Q(x_0, K, T) - P(x_0, K, T)|/|P(x_0, K, T)|\,.$$

1.3.38 Some Numerical Results - Data Description

We consider the daily observed values of the U.S.A. five-Year Interest Rate Swap (see Figure (a)), the corresponding futures prices having maturity September 30th, 2011 (the ticker DSU1 in Figure (b)) and the prices of the corresponding European call and put options with expiry date September 19th, 2011 and strike prices $K_i = 106 + 0.5 * (i - 1)$, $i = 1, 2, \ldots, 18$, (i.e.: $n_C = n_P = 18$) in the period September 14th, 2010, July 20th, 2011. The prices considered are expressed in USD, are daily prices and are the closing price of the day. The strike prices K_i, $i = 1, 2, \ldots, 18$, are expressed in base points. We consider two dates the first one $\hat{t}_1 = October\, 12th, 2010$ selected in a period where the oscillations of futures price are small and the second one $\hat{t}_2 = November\, 15th, 2010$ at the beginning of the fall of the futures prices. Note that from November 12th, 2010 to December 15th, 2010 the futures price goes from the value of 110 to the value of 104 base points.

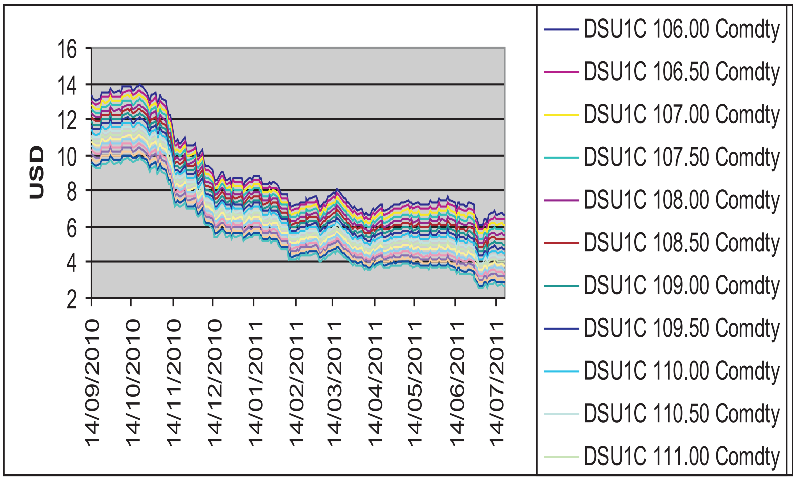

USD
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
14/09/2010
14/10/2010
14/11/2010
14/12/2010
14/01/2011
14/02/2011
14/03/2011
14/04/2011
14/05/2011
14/06/2011
14/07/2011
USSWAP5 Curncy

(a)

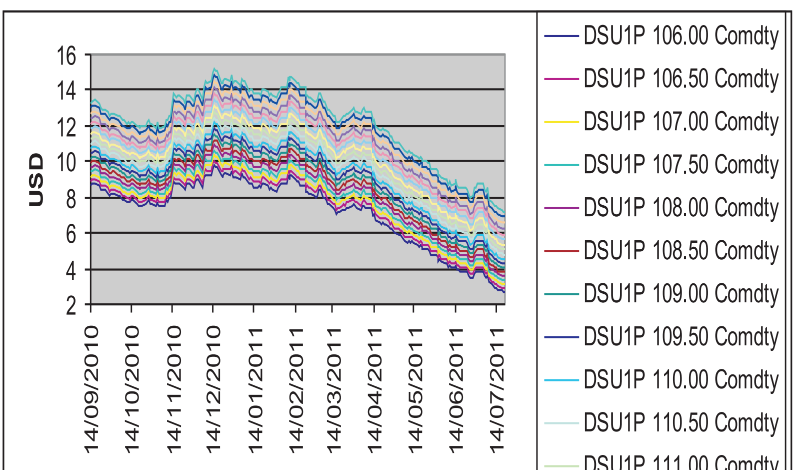

BASE POINTS
114
112
110
108
106
104
102
14/09/2010
14/10/2010
14/11/2010
14/12/2010
14/01/2011
14/02/2011
14/03/2011
14/04/2011
14/05/2011
14/06/2011
14/07/2011
DSU1 Comdty

(b)

1.3.39 Observed European Call and Put Option Prices September 2010 - July 2011

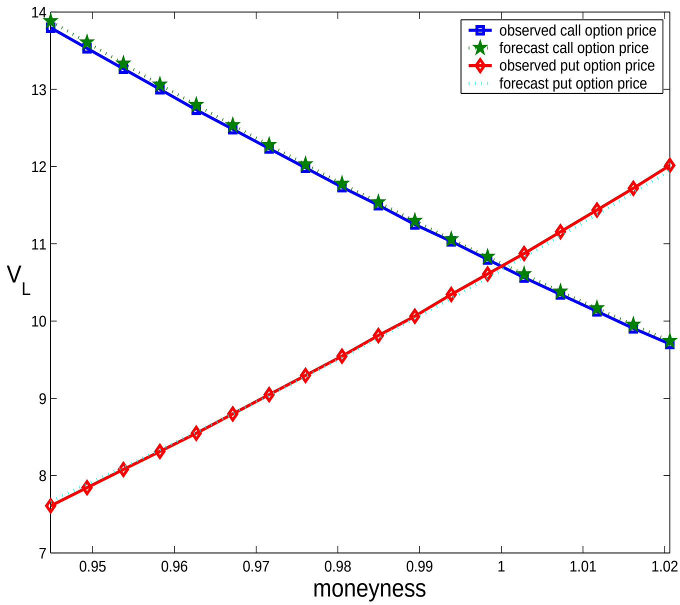


Call option prices on DSU1 with strike price $K_i = 106 + 0.5 * (i - 1)$, $i = 1, 2, \ldots, 18$, and expiry date T= September 19th, 2011 versus time.

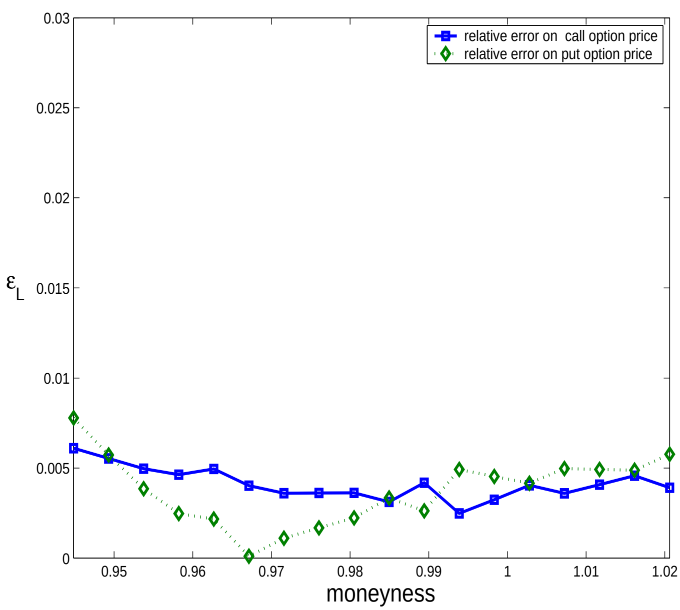


Put option prices on DSU1 with strike price $K_i = 106 + 0.5 * (i - 1)$, $i = 1, 2, \ldots, 18$, and expiry date T= September 19th, 2011 versus time.

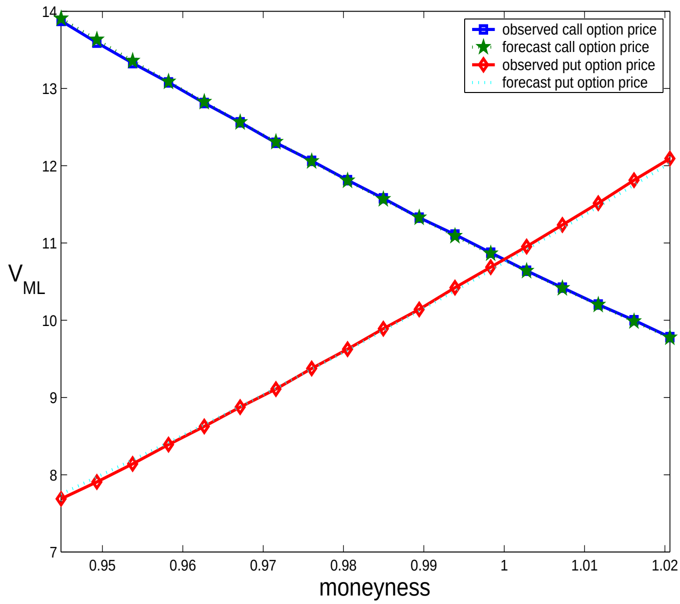
1.3.40 Performance in Predicting Option Prices of the Lognormal Models One Day Ahead



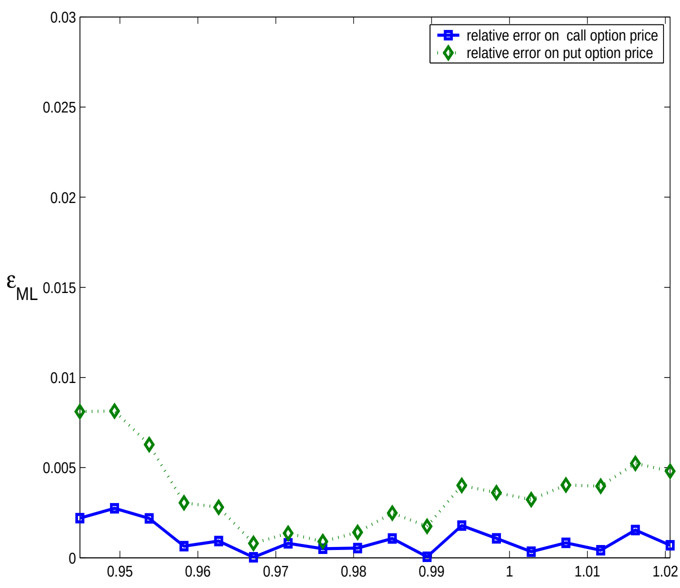
(a)



(b)

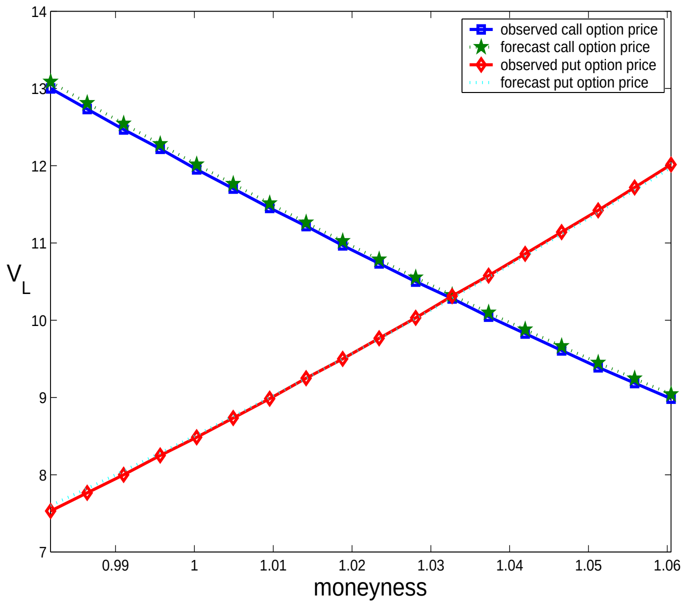


(c)

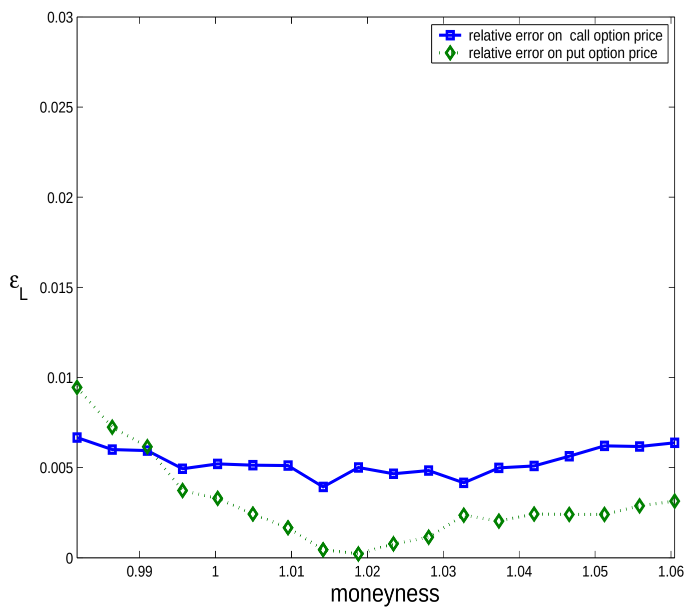


(d)

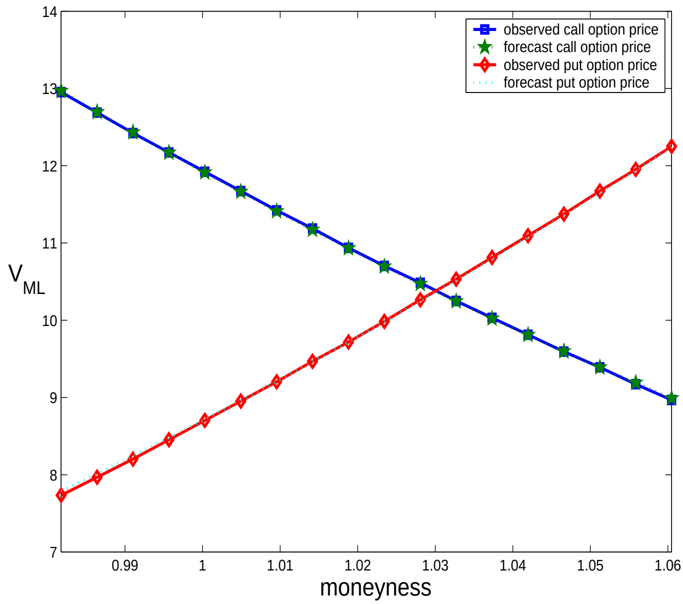
Observed and forecast prices one day in the future of call and put options of lognormal SABR (a) and multiscale SABR (c) and relative errors (b) (lognormal) (d) (lognormal multiscale) obtained calibrating the models at $\hat{t} = t_1 = October\ 12th, 2010$ versus moneyness.


observed call option price
forecast call option price
observed put option price
forecast put option price
V_L
moneyness

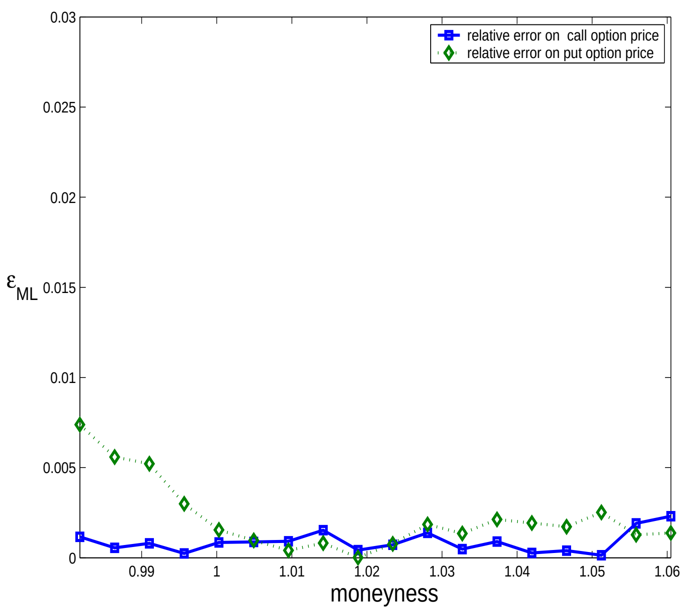
(a)


relative error on call option price
relative error on put option price
ε_L
moneyness

(b)



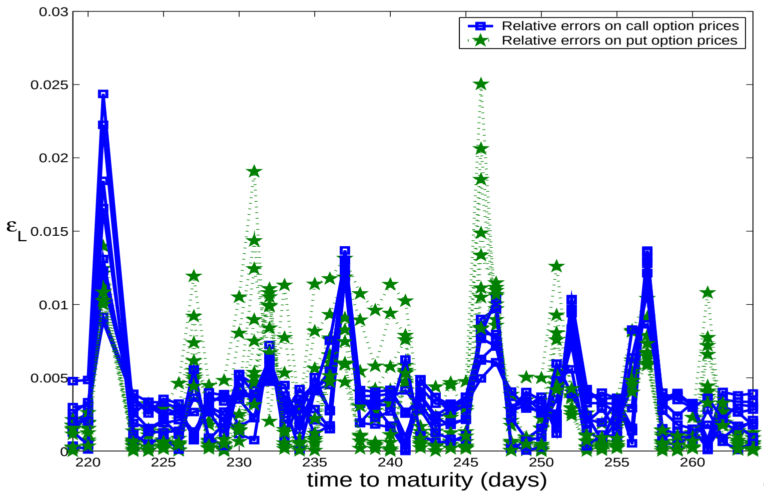
(c)



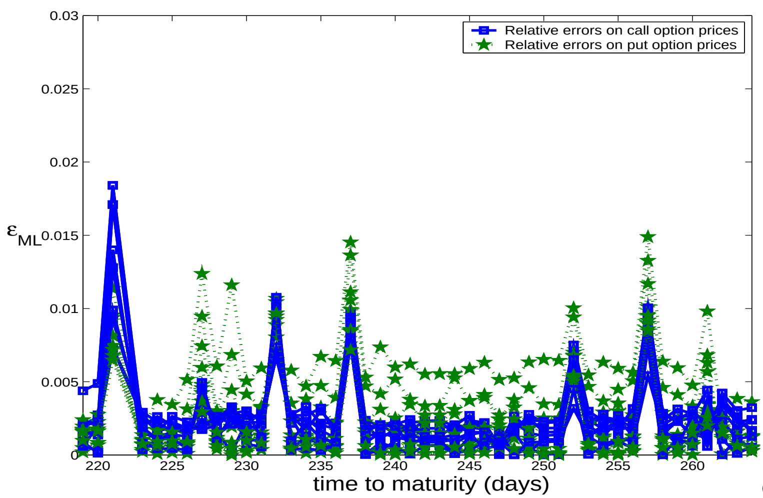
(d)

Observed and forecast prices one day in the future of call and put options of lognormal SABR (a) and multiscale SABR (c) and relative errors (b) (lognormal) (d) (lognormal multiscale) obtained calibrating the models at $\hat{t} = t_2 = November\ 15th, 2010$ versus moneyness.

1.3.41 Performance in Predicting Option Prices of the Lognormal Models One Day Ahead for a Period of Two Months



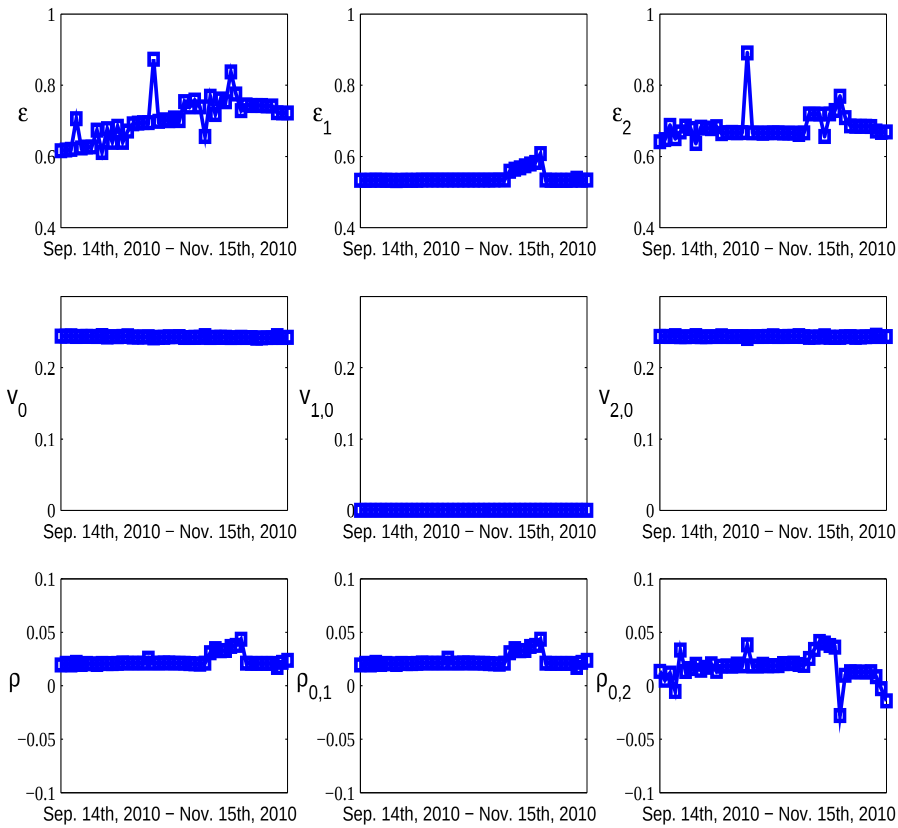
(a)



(b)

Relative errors on the forecast prices one day in the future of call and put options obtained using lognormal SABR model (a) and lognormal multiscale SABR model (b) versus time to maturity expressed in days. The period considered goes from September 14th, 2010, to November 15th, 2010 (interest rate swap experiment).

1.3.42 Model Parameters - Lognormal SABR and Multiscale SABR Models



Parameter values obtained calibrating the lognormal SABR and multiscale SABR models every day for two months in the period going from September 14th, 2010 to November 15th, 2010 versus time (interest rate swap experiment).

The numerical experiments presented show that in the forecast of European option prices the lognormal multiscale SABR model outperforms the lognormal SABR model.

The improvement depends on the set of data analyzed and in some cases may be not significant.

The improvements are more significant when the underlying asset prices present abrupt changes.That is in the situations where the multiscale model that contains a two factor volatility model is expected to outperform the standard model that is based on a one factor volatility model.

1.3.43 Future Work

- Analysis of the SABR and multiscale SABR models with $\beta = 1/2$.
- Asymptotic expansions of the probability density function associated with the state variables of the multiscale SABR model when $\beta \in (0, 1)$ and $\beta \neq 1/2$.
- Use the differential geometry of the models to obtain formulae and numerical algorithms of practical use in mathematical finance.
- Use the "target tracking procedure" presented in [3] for the Heston model to explore the forecasting ability of the SABR and Hull and White models.
- Develop ad hoc statistical tests to calibrate the SABR and Hull and White models.

A general reference to the work in mathematical finance of the authors and of their coauthors is the website:
http://www.econ.univpm.it/recchioni/finance .

1.3.44 References

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