

Research Seminars in **Mathematical Finance**

Stochastic Volatility Models,
Option Pricing,
Calibration



Lorella Fatone
Francesca Mariani
Maria Cristina Recchioni
Francesco Zirilli

Research Seminars in Mathematical Finance: Stochastic Volatility Models, Option Pricing, Calibration

Lorella Fatone

Francesca Mariani

Maria Cristina Recchioni

Francesco Zirilli



Science Publishing Group

548 Fashion Avenue

New York, NY 10018

<http://www.sciencepublishinggroup.com>

Published by Science Publishing Group 2014

Copyright © Lorella Fatone 2014

Copyright © Francesca Mariani 2014

Copyright © Maria Cristina Recchioni 2014

Copyright © Francesco Zirilli 2014

All rights reserved.

First Edition

ISBN: 978-1-940366-02-9

This work is licensed under the Creative Commons
Attribution-ShareAlike 3.0 Unported License. To view a copy of this
license, visit

<http://creativecommons.org/licenses/by-nc/3.0/>



or send a letter to:

Creative Commons

171 Second Street, Suite 300

San Francisco, California 94105

USA

To order additional copies of this book, please contact:

Science Publishing Group

service@sciencepublishinggroup.com

<http://www.sciencepublishinggroup.com>

Printed and bound in India

Preface

This ebook contains a set of slides that can be used to present lectures in a graduate course or in a research seminar in mathematical finance.

The ebook is divided in three chapters, each chapter is divided in three sections, each section contains a set of slides that can be used to present a lecture. The lectures are independent one from the others.

The lectures discuss the results of the research in mathematical finance of the authors during the years 2007-2012. The results presented are concerned with the study of stochastic volatility models (Heston, SABR models and their generalizations), and in particular with the option pricing and calibration problems relative to these models.

These results have been published in papers appeared in academic journals. The papers are complemented with multimedia material and mathematical software available on the web.

To each lecture are associated:

1. a research paper,
2. a website,
3. a set of slides.

The papers and the websites are listed at the beginning of the sections and are already of public domain. This ebook makes of public domain the slides.

A general reference to the work of the authors and of their co-authors in mathematical finance is the website: <http://www.econ.univpm.it/recchioni/finance>.

Contents

| | | |
|----------|--|----------|
| 1 | Stochastic Volatility Models | 1 |
| 1.1 | Heston Model: Option Pricing Formulae, Filtering and Calibration Problems | 3 |
| 1.1.1 | Outline of the Presentation | 4 |
| 1.1.2 | The Calibration Problem for the Heston Stochastic Volatility Model | 4 |
| 1.1.3 | The Heston Stochastic Volatility Model | 5 |
| 1.1.4 | The Calibration Problem | 6 |
| 1.1.5 | Solution of the Calibration Problem | 8 |
| 1.1.6 | The Filtering Problem | 10 |
| 1.1.7 | Some Numerical Results | 14 |
| 1.1.8 | References | 20 |
| 1.2 | Multiscale Heston Model: Option Pricing Formulae and Calibration Problems | 22 |

| | | |
|--------|--|----|
| 1.2.1 | Outline of the Presentation | 23 |
| 1.2.2 | Calibration and Filtering Problems for a Stochastic Volatility Model | 23 |
| 1.2.3 | Heston Stochastic Volatility Model | 25 |
| 1.2.4 | Heston Stochastic Volatility Model (More) | 25 |
| 1.2.5 | Heston Model Calibration: Numerical Results on Real Data . . . | 26 |
| 1.2.6 | Heston Model Calibration: Numerical Results on Real Data (More) | 27 |
| 1.2.7 | Heston Model: Conclusion | 28 |
| 1.2.8 | The Multiscale Stochastic Volatility Model | 29 |
| 1.2.9 | Why Using a Multiscale Model? | 30 |
| 1.2.10 | The Multiscale Stochastic Volatility Model | 31 |
| 1.2.11 | Spikes Using Multiscale Stochastic Volatility Model | 32 |
| 1.2.12 | Transition Probability Density Function | 33 |
| 1.2.13 | One Dimensional Integral Formula for Transition Probability Density Function | 34 |
| 1.2.14 | The Elementary Functions Defining the Transition Probability Density Function | 35 |
| 1.2.15 | European Vanilla Call Option | 36 |
| 1.2.16 | European Vanilla Put Option | 37 |
| 1.2.17 | Calibration Problem | 37 |
| 1.2.18 | Numerical Results on Synthetic Data | 38 |
| 1.2.19 | Numerical Results on Real Data: S&P 500 Index | 39 |
| 1.2.20 | Forecasted Values of Call and Put Options | 40 |

| | |
|--|----|
| 1.2.21 Absolute Errors on Forecasted Values of Call Option Prices Heston-Multiscale Model | 41 |
| 1.2.22 Future Work | 41 |
| 1.2.23 References | 42 |
| 1.3 SABR and Multiscale SABR Models: Option Pricing and Calibration | 43 |
| 1.3.1 Outline of the Presentation | 44 |
| 1.3.2 Notations | 44 |
| 1.3.3 Motivations and Background | 45 |
| 1.3.4 SABR Model | 45 |
| 1.3.5 Multiscale SABR Model | 46 |
| 1.3.6 Multiscale SABR Model (cont.) | 47 |
| 1.3.7 Hull and White Stochastic Volatility Model | 48 |
| 1.3.8 Some Open Problems about the SABR and the Hull and White Models | 49 |
| 1.3.9 Last Year Results | 49 |
| 1.3.10 This Year Results on SABR and Hull and White Models and on Their Multiscale Versions | 50 |
| 1.3.11 This Year Results on SABR and Hull and White Models and on Their Multiscale Versions (cont.) | 51 |
| 1.3.12 The Heat Kernel: An Example | 52 |
| 1.3.13 Solution Procedure | 53 |
| 1.3.14 The Heat Kernel - Step 1 | 53 |

| | | |
|--------|--|----|
| 1.3.15 | The Heat Kernel - Step 2-3 | 54 |
| 1.3.16 | The Heat Kernel - Step 3 | 55 |
| 1.3.17 | The Heat Kernel | 56 |
| 1.3.18 | The Transforms Corresponding to the Kernels Used in the Study of the SABR and the Hull and White Models | 56 |
| 1.3.19 | Resolutions of the Identity Associated to the Previous Integral Transforms | 57 |
| 1.3.20 | This Year Results on SABR and Hull and White Models (cont.) . | 58 |
| 1.3.21 | This Year Results on SABR and Hull and White Models | 58 |
| 1.3.22 | SABR Backward Kolmogorov Equation | 59 |
| 1.3.23 | Solution Procedure | 60 |
| 1.3.24 | The Heat Kernel of SABR Model $\beta = 0, \beta = 1$ | 61 |
| 1.3.25 | The Heat Kernel of SABR Model $\beta = 0, \beta = 1$ (cont.) | 61 |
| 1.3.26 | The Joint Transition Probability Density Function of SABR Model $\beta = 0, \beta = 1$ (cont.) | 63 |
| 1.3.27 | Option Pricing in the SABR Model $\beta = 0, \beta = 1$ | 64 |
| 1.3.28 | European Call Option Price in the SABR and Multiscale SABR Models with $\beta = 0, 1$ | 64 |
| 1.3.29 | The Joint Transition Probability Density Function of Hull and White Model | 65 |
| 1.3.30 | The Heat Kernel of the SABR Model $\beta \in (0, 1), \rho = 0$ | 66 |
| 1.3.31 | Riemannian Metric Associated to the SABR Models $\beta \in [0, 1)$. . | 67 |
| 1.3.32 | Geodesics Equations Associated to the SABR Models $\beta \in [0, 1)$. | 68 |

| | |
|--|-----------|
| 1.3.33 Geodesics Associated to the SABR Models $\beta \in [0, 1)$ | 69 |
| 1.3.34 Geodesic Curves Associated to the SABR Models $\beta \in [0, 1)$ | 70 |
| 1.3.35 Geodesic Curves Associated to the SABR Models $\beta \in [0, 1)$ | 70 |
| 1.3.36 Calibration Problem | 71 |
| 1.3.37 Formulation of the Calibration Problem | 72 |
| 1.3.38 Some Numerical Results - Data Description | 72 |
| 1.3.39 Observed European Call and Put Option Prices September 2010 - July 2011 | 75 |
| 1.3.40 Performance in Predicting Option Prices of the Lognormal Models One Day Ahead | 76 |
| 1.3.41 Performance in Predicting Option Prices of the Lognormal Models One Day Ahead for a Period of Two Months | 80 |
| 1.3.42 Model Parameters - Lognormal SABR and Multiscale SABR Models | 81 |
| 1.3.43 Future Work | 82 |
| 1.3.44 References | 82 |
| 2 The Study of Real Data Using ad hoc Stochastic Models | 85 |
| 2.1 Maximum Likelihood Estimation of the Parameters of a Stochastic Differential System Modeling the Returns of the Index of Some Classes of Hedge Funds | 87 |
| 2.1.1 Outline of the Presentation | 88 |
| 2.1.2 “Long/Short Equity” Hedge Funds | 88 |

| | | |
|--------|---|-----|
| 2.1.3 | A Single Factor Model for the Index of “Long/Short Equity” Hedge Funds | 89 |
| 2.1.4 | From a Discrete Time Model to a Continuous Time Model | 89 |
| 2.1.5 | The Stochastic Model for “Long/Short Equity” Hedge Funds . . . | 90 |
| 2.1.6 | Forecasting and Estimation Problems | 91 |
| 2.1.7 | Filtering Problem | 92 |
| 2.1.8 | Solution of the Filtering Problem | 92 |
| 2.1.9 | Integral Representation Formula for the Fundamental Solution . . | 93 |
| 2.1.10 | Integral Representation Formulae for the Forecasted Values . . . | 95 |
| 2.1.11 | Forecasted Values of x_t, z_t, v_t | 96 |
| 2.1.12 | Estimation Problem | 97 |
| 2.1.13 | Numerical Results on Synthetic Data | 98 |
| 2.1.14 | Analysis of a Two Years Time Series | 99 |
| 2.1.15 | Numerical Results on Real Data (Banca Akros, Milano) | 100 |
| 2.1.16 | Is the Model for the Return of the Index of the Hedge Funds a Satisfactory Model? | 102 |
| 2.1.17 | Comparison Between Forecasted Values and Data of the Hedge Fund Index Returns | 103 |
| 2.1.18 | Future Work | 103 |
| 2.1.19 | References | 104 |
| 2.2 | Calibration of a Stochastic Model of Spiky Prices: An Application to Electric Power Prices | 105 |
| 2.2.1 | The Calibration Problem | 106 |

| | | |
|--------|--|-----|
| 2.2.2 | The Model for Spiky Asset Prices | 107 |
| 2.2.3 | The Process for Asset Prices with Spikes | 111 |
| 2.2.4 | The Calibration Problem (More) | 114 |
| 2.2.5 | Solution of the Calibration Problem | 115 |
| 2.2.6 | The Filtering Problem | 117 |
| 2.2.7 | Some Numerical Results on Real Data | 119 |
| 2.2.8 | Is the Model for Spiky Prices a Satisfactory Model? | 121 |
| 2.2.9 | Comparison Between the Real and the Forecasted Forward Electric Power Prices | 123 |
| 2.2.10 | Month-Ahead Prices | 124 |
| 2.2.11 | Quarter-Ahead Prices | 125 |
| 2.2.12 | 1 Year-Ahead Prices | 126 |
| 2.2.13 | References | 126 |
| 2.3 | The Analysis of Electric Power Price Data and of the S&P 500 Index Using a Multiscale Stochastic Volatility Model | 128 |
| 2.3.1 | Outline of the Presentation | 129 |
| 2.3.2 | Notations | 129 |
| 2.3.3 | Electric Power Prices and Associated Log-Return | 130 |
| 2.3.4 | The Multiscale Stochastic Volatility Model | 131 |
| 2.3.5 | The Multiscale Stochastic Volatility Model Generalizes the Heston Model | 132 |
| 2.3.6 | Model Parameters | 132 |

| | | |
|--------|---|-----|
| 2.3.7 | Why do We Use a Multiscale Model? | 133 |
| 2.3.8 | Spikes Generated by the Multiscale Stochastic Volatility Model | 134 |
| 2.3.9 | An Explicitly Solvable Model | 135 |
| 2.3.10 | One Dimensional Integral Formula for the Transition Probability Density Function of the Multiscale Model | 135 |
| 2.3.11 | The Elementary Functions Appearing in the Transition Probability Density Function | 136 |
| 2.3.12 | European Vanilla Call Option Price in the Multiscale Stochastic Volatility Model | 136 |
| 2.3.13 | Conditioned Joint Probability Density Function | 137 |
| 2.3.14 | The Functions $f_i, i = 0, 1, \dots, n$ | 138 |
| 2.3.15 | Formulae to Forecast the Log-Return and the Associated Variances | 139 |
| 2.3.16 | Calibration and Filtering Problems | 139 |
| 2.3.17 | Calibration Problem - Maximum Likelihood Approach (ML) | 140 |
| 2.3.18 | Calibration Problem - Least Squares Approach (LS) | 141 |
| 2.3.19 | Numerical Results on Real Data: S&P 500 Index | 142 |
| 2.3.20 | Forecasted Values of Call and Put Options | 143 |
| 2.3.21 | Maximum Likelihood Calibration - Errors on Forecasts | 144 |
| 2.3.22 | Comparison Between Least Squares (LS) Approach and Maximum Likelihood (ML) Approach | 144 |
| 2.3.23 | Analysis of Electric Power Prices | 145 |
| 2.3.24 | Analysis of Electric Power Prices: Numerical Results | 148 |
| 2.3.25 | Analysis of Electric Power Prices: Description of the Results | 151 |

| | | |
|----------|--|------------|
| 2.3.26 | Analysis of Electric Power Prices: Forecasted Prices | 154 |
| 2.3.27 | Future Work | 156 |
| 2.3.28 | References | 156 |
| 3 | Calibration of Stochastic Volatility Models Using Statistical Tests | 159 |
| 3.1 | The Use of Statistical Tests to Calibrate the Black-Scholes Model | 161 |
| 3.1.1 | Outline of the Presentation | 162 |
| 3.1.2 | The Calibration Problem | 162 |
| 3.1.3 | The Calibration Problem for the Black-Scholes Model | 164 |
| 3.1.4 | The Calibration Problem | 166 |
| 3.1.5 | The Student's T Test and the χ^2 Test for the Gaussian Random Variable | 167 |
| 3.1.6 | The Student's T Test | 167 |
| 3.1.7 | The χ^2 Test | 168 |
| 3.1.8 | Large Option Prices with Uncertain Volatility and Statistical Significance | 170 |
| 3.1.9 | Some Numerical Results on Real Data | 174 |
| 3.1.10 | The <i>S&P</i> 500 Index (year 2005) | 175 |
| 3.1.11 | The <i>S&P</i> 500 Daily Log-Return Increments (Year 2005) | 176 |
| 3.1.12 | The Parameters σ^2 and μ Reconstructed from the 2005 <i>S&P</i> 500 Data | 179 |
| 3.1.13 | References | 185 |
| 3.2 | The Use of Statistical Tests to Calibrate the Normal SABR Model I | 186 |

| | | |
|--------|--|-----|
| 3.2.1 | Outline of the Presentation | 187 |
| 3.2.2 | Statistical Test Review | 188 |
| 3.2.3 | Statistical Test - Hypothesis Testing | 188 |
| 3.2.4 | Elementary Statistical Tests | 189 |
| 3.2.5 | Black-Scholes vs Normal SABR Model | 190 |
| 3.2.6 | The Black-Scholes Model | 191 |
| 3.2.7 | The Black-Scholes Calibration Problem | 192 |
| 3.2.8 | The Normal SABR Model | 193 |
| 3.2.9 | The Normal SABR Model Interpreted as a State Space Model . . | 195 |
| 3.2.10 | The Normal SABR Calibration Problem | 198 |
| 3.2.11 | Procedure Used to Determine the Threshold $r_{\alpha,n}$ | 201 |
| 3.2.12 | A Drawback | 202 |
| 3.2.13 | A Numerical Example | 203 |
| 3.2.14 | Future Work | 207 |
| 3.2.15 | References | 207 |
| 3.3 | The Use of Statistical Tests to Calibrate the Normal SABR Model II . . . | 209 |
| 3.3.1 | Outline of the Presentation | 210 |
| 3.3.2 | Normal SABR Model | 210 |
| 3.3.3 | Calibration Techniques | 213 |
| 3.3.4 | An ad hoc Statistical Test | 214 |
| 3.3.5 | Hypothesis Formulation | 214 |

| | | |
|--------|---|-----|
| 3.3.6 | Decision Rule | 215 |
| 3.3.7 | The Test Statistic | 215 |
| 3.3.8 | Monte Carlo Method | 217 |
| 3.3.9 | Importance Sampling | 218 |
| 3.3.10 | Ad hoc Statistical Test (more) | 219 |
| 3.3.11 | Sampling Distribution | 220 |
| 3.3.12 | Threshold Value | 220 |
| 3.3.13 | Decision Rule | 221 |
| 3.3.14 | An Example Using Synthetic Data | 221 |
| 3.3.15 | References | 223 |

